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Highly rotating fluids in rough domains

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Abstract

We consider a rotating fluid in a domain with rough horizontal boundaries. The Rossby number, kinematic viscosity and roughness are supposed of characteristic size ε. We prove a strong convergence theorem on solutions of Navier–Stokes–Coriolis equations, as ε goes to 0, in the well-prepared case. We show in particular that the limit system is a two-dimensional Euler equation with a nonlinear damping term due to boundary layers. We thus give a substantial refinement of the results obtained on flat boundaries with the classical Ekman layers.

Résumé

On étudie dans cet article le système des fluides tournants, dans un domaine limité par deux parois horizontales irrégulières. Le nombre de Rossby, la viscosité et la taille caractéristique de la rugosité sont supposés du même ordre ε. On montre la convergence forte des solutions de ce système quand ε tend vers 0, pour des données initiales bien-préparées. En particulier, on montre que le système limite est une équation d'Euler 2D, avec un terme d'amortissement non-linéaire dû aux couches limites situées près des bords du domaine. Ce résultat améliore ainsi substantiellement ceux obtenus dans le cas de parois planes, avec les traditionnelles couches d'Ekman.

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Keywords: Incompressible fluid mechanics; Rapidly rotating fluids; Rough boundaries

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Introduction

We will study in this paper Navier–Stokes–Coriolis equations:

$$\partial_t u + u \cdot \nabla u + \frac{\mathbf{e} \times u}{\varepsilon} + \frac{\nabla p}{\varepsilon} - \nu \Delta u = 0, \tag{0.1}$$

$$\nabla \cdot u = 0. \tag{0.2}$$

This system models the evolution of an incompressible rotating fluid, submitted to a Coriolis force $\varepsilon^{-1}\mathbf{e} \times u$ and viscous forces $-\nu \Delta u$. Vector $\mathbf{e} = (0, 0, 1)^t$ is the rotation axis. Parameters ε and ν are respectively the Rossby number and the kinematic viscosity. In the sequel, we will suppose $\nu = \varepsilon \ll 1$. It is a geophysical scaling, notably relevant to the Earth's liquid core, for which $\nu \sim 10^{-8}$ and $\varepsilon \sim 10^{-7}$. We refer to [8] for more details.

System (0.1), (0.2) then reduces to

$$\partial_t u + u \cdot \nabla u + \frac{\mathbf{e} \times u}{\varepsilon} + \frac{\nabla p}{\varepsilon} - \varepsilon \Delta u = 0, \tag{0.3}$$

$$\nabla \cdot u = 0, \tag{0.4}$$

in a domain Ω^ε to be precised later on. Completed with appropriate initial data and boundary conditions, this system has global Leray solutions (see [19]),

$$u^\varepsilon \in L^\infty\big(0, +\infty;\ L^2\big)^3 \cap L^2\big(0, +\infty;\ H^1\big)^3.$$

The proof is the same as for classical Navier–Stokes equations, because the Coriolis term does not play any role in the energy estimates. It is then natural to ask about the behaviour of u^ε as ε goes to 0.

In the case of flat boundaries (for instance $\Omega^\varepsilon = \Omega = \mathbb{T}^2 \times (0, 1)$), the situation has been widely studied from both physical and mathematical points of view. One can construct an approximate solution of type

$$u^\varepsilon_{\text{app}}(t, x, y, z) = u(t, x, y) + \tilde{u}\left(t, x, y, \frac{z}{\varepsilon}\right) + \bar{u}\left(t, x, y, \frac{1-z}{\varepsilon}\right),$$

where

- u is a two-dimensional interior term (i.e., $u_3 = 0$).
- $\tilde{u} = \tilde{u}(t, x, y, \theta)$ and $\breve{u} = \breve{u}(t, x, y, \lambda)$ are boundary layer terms (Ekman layers), solutions of a linear differential system in θ and λ, respectively.

Under suitable assumptions, u^ε converges to u in $L^\infty(0, +\infty;\ L^2)^3$. The limit term u is the solution of a damped Euler equation, with a dissipative term due to the boundary layer (the so-called "Ekman pumping"). For physical background, see textbooks [12] or [18]. For mathematical work on Ekman layers, we refer to [5,6,13].

The aim of this paper is to extend these results to boundaries with irregularities. We will consider irregularities with characteristic size ε, in both horizontal and vertical directions. This problem has serious physical motivation: the liquid core of the Earth is thought to be responsible for the geomagnetic field through self excited dynamo action. It meets the solid mantle some 3000 km below the surface. A boundary layer developps there, which is strongly influenced by rotation, thus of the Ekman type (see [7]). This layer is central in most asymptotic dynamo models (see [15] for a discussion). Recent modeling showed that the core-mantle boundary is in fact rough with a typical scale (both height and wavelength) comparable to the boundary layer width ε, i.e., of the order of a meter [17]. It is thus important from a physical point of view to understand how this roughness affects the layer and its stability.

Mathematically, the effect of rugosity on a flow has been widely studied in the context of wall laws: see for instance articles [1,2,9,14] and references therein. Most of these papers study channel flows, for which the boundary layer correction to the limit flow (for instance Poiseuille or Couette flow) has at most amplitude $\mathrm{O}(\varepsilon)$. In the system (0.3), (0.4) studied here, boundary layers have an amplitude $\mathrm{O}(1)$ and modify the limit flow. Moreover, the equations involved in the construction of the approximate solution will be much more complex than in the “flat case”: the differential system on the boundary layer will turn into a nonlinear PDE, and the Ekman pumping term also becomes nonlinear.

The rest of this paper will be structured as follows. In Section 1, we describe precisely the rough domain and state the main result of our paper. In Section 2, we construct formally an approximate solution of u^ε, and identify equations satisfied by both the boundary layer and the interior parts of the approximation. In Section 3, we solve the boundary layer system. In Section 4, we solve the interior system. We end in Section 5 with the convergence theorem, and discuss possible extensions and open questions.

1. Description of the domain and statement of the results

1.1. The domain Ω^ε

Let us model the domain Ω^ε in which Eqs. (0.3), (0.4) hold. As in [14], we shall write:

$$\Omega^\varepsilon = \Omega \cup \Sigma_1 \cup \Sigma_2 \cup \Omega_1^\varepsilon \cup \Omega_2^\varepsilon.$$

- Ω is the interior domain $(0,1)^3$.
- $\Sigma_1 = (0,1)^2 \times \{0\}$ and $\Sigma_2 = (0,1)^2 \times \{1\}$ are the interfaces.
- Ω_1^ε and Ω_2^ε are the rough layers. They are supposed to be generated by homothety and translations of “canonical cells of roughness”.

More precisely, for $j = 1, 2$, let γ_j a Lipschitz surface, $Z = \gamma_j(X, Y)$, $\gamma_j : (0,1)^2 \mapsto [0,1)$. We assume that $\Gamma_j = \bigcup_{k\in\mathbb{Z}^2}(k + \gamma_j)$ is also a Lipschitz surface. The canonical cells of roughness are defined by:

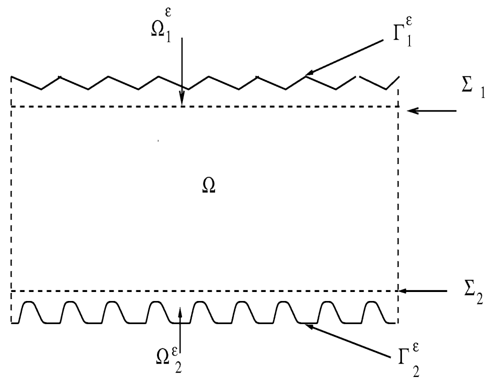

Fig. 1. The domain Ω^{ε}.

$$\mathcal{R}_j = \big\{(X, Y, Z) \,\big|\, (X, Y) \in (0, 1)^2,\ 1 > Z > \gamma_j(X, Y)\big\}.$$

We then set:

$$\Omega_1^{\varepsilon} = \left(\bigcup_{k \in \mathbb{Z}^2} \varepsilon\big(\mathcal{R}_1 + (k_1, k_2, -1)\big)\right) \cap \big((0, 1)^2 \times (-\varepsilon, 0)\big).$$

For simplicity, we suppose that $1/\varepsilon$ is an integer, so that Ω_1^{ε} consists of a large number of periodically distributed humps of characteristic length and amplitude ε.

In the same way ($\mathbf{e} = (0, 0, 1)^t$),

$$\Omega_2^{\varepsilon} = \mathbf{e} + \left(\bigcup_{k \in \mathbb{Z}^2} \varepsilon\big(-\mathcal{R}_2 + (k_1, k_2, 1)\big)\right) \cap \big((0, 1)^2 \times (1, 1+\varepsilon)\big).$$

We note Γ_1^{ε} and Γ_2^{ε} the lower and upper horizontal boundaries of Ω^{ε}.

Remark. We could have chosen a different characteristic length for upper regularities, $\varepsilon' = C\varepsilon$, assuming for simplicity that $1/\varepsilon' \in \mathbb{N}$.

In what follows, we will consider solutions of (0.3), (0.4) satisfying:

$$u^{\varepsilon} \text{ 1-periodic in } (x, y), \quad u^{\varepsilon} = 0 \quad \text{at } \Gamma_1^{\varepsilon} \cup \Gamma_2^{\varepsilon}. \tag{1.1}$$

For this boundary condition, the existence of Leray solutions is of course still valid.

Besides the global domain Ω^{ε}, we also need to introduce boundary layer domains $\widetilde{\omega}$ and $\overline{\omega}$ defined as follows:

$$\widetilde{\omega} = \widetilde{\omega}^{+} \cup \Sigma \cup \widetilde{\omega}^{-}, \qquad \overline{\omega} = \overline{\omega}^{+} \cup \Sigma \cup \overline{\omega}^{-},$$

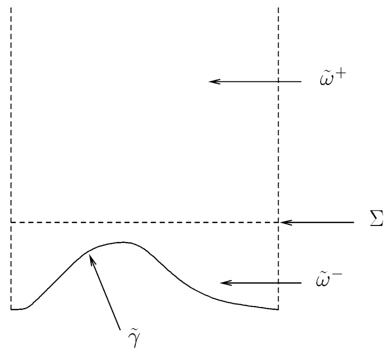

Fig. 2. The lower boundary layer domain $\widetilde{\omega}$.

where

$$\widetilde{\omega}^+ = (0,1)^2 \times \mathbb{R}^+, \qquad \Sigma = (0,1)^2 \times 0, \qquad \widetilde{\omega}^- = R_1 - (0,0,1),$$
$$\overline{\omega}^+ = -R_2 + (0,0,1), \qquad \overline{\omega}^- = (0,1)^2 \times \mathbb{R}^-.$$

We note $\tilde{\gamma}$ and $\bar{\gamma}$ the horizontal boundaries of $\widetilde{\omega}$ and $\overline{\omega}$. Finally, for all positive R, R_1 and R_2, we note:

$$\widetilde{\omega}^R = \widetilde{\omega} \cap \{z > R\}, \qquad \widetilde{\omega}^{R_1,R_2} = \widetilde{\Omega} \cap \{R_1 < Z < R_2\}.$$

1.2. Statement of the main results

As usual with boundary layer problems, the study of u^ε involves auxiliary systems.

- The first one (which is the boundary layer system) holds in $\widetilde{\omega}$: for $\mathbf{u} \in \mathbb{R}^2$, we consider equations:

$$\begin{aligned}
&\mathbf{e} \times \tilde{u} + \nabla \tilde{p} + \tilde{u} \cdot \nabla \tilde{u} - \Delta \tilde{u} = \begin{pmatrix} -\mathbf{u}^\perp \\ 0 \end{pmatrix} \quad \text{in } \widetilde{\omega}^-, \\
&\mathbf{e} \times \tilde{u} + \nabla \tilde{p} + \tilde{u} \cdot \nabla \tilde{u} - \Delta \tilde{u} = 0 \quad \text{in } \widetilde{\omega}^+, \\
&\nabla \cdot \tilde{u} = 0 \quad \text{in } \widetilde{\omega}^+ \cup \widetilde{\omega}^-, \\
&[\tilde{u}]|_\Sigma = -\begin{pmatrix} \mathbf{u} \\ 0 \end{pmatrix} \quad \text{on } \Sigma, \\
&\left[\frac{\partial \tilde{u}}{\partial Z} - \tilde{p}\mathbf{e}\right]\Big|_\Sigma = 0 \quad \text{on } \Sigma, \\
&\tilde{u} = 0 \quad \text{on } \tilde{\gamma}, \ \tilde{u} \text{ 1-periodic in } (X, Y),
\end{aligned} \tag{BL}$$

where $\tilde{u}:\widetilde{\omega}\mapsto\mathbb{R}^3$, $\tilde{p}:\widetilde{\omega}\mapsto\mathbb{R}$, $\mathbf{e}=(0,0,1)$, and $[f]|_\Sigma = f^+ - f^-$ is the jump of f at the interface Σ. We prove in Section 3

Theorem 1.1. *There exists $U_\infty\in\mathbb{R}$, such that for all $|\mathbf{u}|\leqslant U_\infty$,* (BL) *has a unique variational solution $(\tilde{u},\tilde{p})$, in the sense given in Section* 3.

Moreover, for R large enough, for all $m\geqslant 0$, $(\tilde{u},\tilde{p})\in H^m(\widetilde{\omega}^R)^4$ with the estimate

$$\|\tilde{u}\|_{H^m(\widetilde{\omega}^R)} + \|\tilde{p}\|_{H^m(\widetilde{\omega}^R)} \leqslant C_m \exp(-\sigma R),$$

where $\sigma>0$ is independent of m, R.

- The second one (which is the limit system) is two-dimensional. It holds in $\mathbb{T}^2$, i.e., $(0,1)^2$ with 1-periodic boundary conditions on (x,y); we consider equations:

$$\begin{aligned}
&\partial_t\zeta + u\cdot\nabla\zeta + \operatorname{curl} P(u) = 0,\\
&\zeta = \operatorname{curl} u, \quad \nabla\cdot u = 0,\\
&u|_{t=0} = u_0,
\end{aligned}\tag{Int}$$

  where $u=u(t,x,y):\mathbb{R}^+\times\mathbb{T}^2\mapsto\mathbb{R}^2$, $\operatorname{curl} u = \partial_x u_2 - \partial_y u_1$. The “pumping function” P is defined by $P=\widetilde{P}+\overline{P}$, where $\widetilde{P}(\mathbf{u}) = \int_{\widetilde{\omega}}\binom{\tilde{u}_2}{-\tilde{u}_1}$, $|\mathbf{u}|<U_\infty$, $\tilde{u}$ solution of (BL), as given by Theorem 1.1. Function $\overline{P}$, related to the upper boundary layer, is similar (see Section 4). $P(\mathbf{u})$ is a dissipative term, as will be shown in:

Proposition 1.2. *Let U_∞ given as in Theorem* 1.1 *if $\mathbf{u}\in\mathbb{R}^2$ satisfies $|\mathbf{u}|\leqslant U_\infty$, then $P(\mathbf{u})\cdot\mathbf{u}\geqslant 0$.*

We introduce the homogeneous Sobolev space:

$$\dot{H}^m(\mathbb{T}^2) = \left\{ w\in H^m(\mathbb{T}^2),\ \int w = 0\right\},\quad m\geqslant 0,$$

and state:

Theorem 1.3. *Let $m\geqslant 3$, $u_0\in\dot{H}^m(\mathbb{T}^2)^2$. There exists $T_m>0$, $\delta_m>0$, such that: if $\|u_0\|_{L^\infty}\leqslant\delta_m$,* (Int) *has a unique strong solution*:

$$u\in\mathcal{C}^0\big([0,T_m];\ \dot{H}^m\big)^2\cap\mathcal{C}^1\big((0,T_m];\ \dot{H}^{m-1}\big)^2.$$

Once these auxiliary systems are solved, we prove the following convergence result:

Theorem 1.4. *Let $u_0\in\dot{H}^3(\mathbb{T}^2)^2$. Let u the associate solution of* (Int). *We define u^0 on Ω^ε by*:

$$u^0 = \begin{pmatrix} u\\ 0\end{pmatrix}\quad in\ \Omega,\qquad u^0 = 0\quad in\ \Omega\backslash\Omega^\varepsilon.$$

There exists $\delta > 0$ *and* $T > 0$ *such that, for any weak solution* u^ε *of* (0.3), (0.4), (1.1),

$$\left(\sup |u_0| \leqslant \delta \text{ and } \left\| \left(u^\varepsilon - u^0\right)(0,\cdot) \right\|_{L^2} \underset{\varepsilon \to 0}{\longrightarrow} 0\right) \quad \Rightarrow \quad \left\| u^\varepsilon - u^0 \right\|_{L^\infty(0,T;L^2)} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

Remark. Let us comment the physical meaning of above results. The 2D nature of the limit flow is a consequence of the so-called geostrophic balance: the pressure gradient compensates the Coriolis force, and the fluid velocity turns to be invariant along the rotation axis. This limit flow solves a modified Euler equation (given in curl form (Int)). In a domain without boundaries, this limit system would have been a classical Euler equation, because of the evanescent viscosity. But in our case, boundary layers create an inflow ("Ekman pumping"), and thus a circulation of fluid, which dissipates energy through friction. Mathematically, this phenomenon is responsible for the additional dissipative term $P(u)$, in link with the boundary layer system (BL). Further comments and questions about this pumping term will be raised in last section.

Remarks. (1) System (BL) is a generalization of the differential system satisfied by the Ekman profile in the case of flat boundaries:

$$\mathbf{e} \times \tilde{u} - \frac{\partial^2 \tilde{u}}{\partial Z^2} = 0, \quad \tilde{u}(0) = -\begin{pmatrix} \mathbf{u} \\ 0 \end{pmatrix}.$$

As in the study of stationary Navier–Stokes equations, a smallness assumption on $|\mathbf{u}|$ is required to make system (BL) well posed. Indeed, as pointed out in [11,19], systems of this type may have two distinct solutions at large "Reynolds number": we refer to [11] for more details.

(2) System (Int) is a generalization of the damped Euler equation satisfied by the interior term in the case of flat boundaries (see [13]),

$$\begin{aligned} &\partial_t \zeta + u \cdot \nabla \zeta + \sqrt{2}\,\zeta = 0, \\ &\zeta = \operatorname{curl} u, \quad \nabla \cdot u = 0, \\ &u|_{t=0} = u_0. \end{aligned} \tag{1.2}$$

In this last equation, the damping term $\sqrt{2}\,\zeta$ leads to a decrease of the L^∞ norm of ζ. As for 2D Euler equations (see [4,16]), the method of Yudovitch applies and yields the existence of global smooth solutions.

For system (Int), we do not manage to get a so good control on operator P, so that we only have the existence of regular solutions for short times. Note that the smallness assumption on $\|u\|_{L^\infty}$ (or equivalently $\|u_0\|_{L^\infty}$) is linked to the solvability of (BL).

(3) The proof of Theorem 1.4 says more than the theorem itself: broadly speaking, it shows that as long as the solution u of (Int) remains smooth and small in L^∞ norm, there is convergence of weak solution u^ε to u (for appropriate initial data). With this formulation, we see that it is an extension of the convergence theorem stated in [7].

2. Formal asymptotic expansion

2.1. Ansatz

2.1.1. Profiles

We wish to construct an approximate solution of (0.3), (0.4), (1.1), made as usual of interior and boundary layer "profiles". We look for an Ansatz of type: for all $t > 0$, for all $(x, y, z) \in \Omega^\varepsilon$,

$$u^\varepsilon_{\mathrm{app}}(t,x,y,z) = \sum_{i=0}^{n} \varepsilon^i \left(u^i(t,x,y,z) + \tilde{u}^i\left(t,x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{z}{\varepsilon}\right) + \bar{u}^i\left(t,x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{z-1}{\varepsilon}\right)\right), \tag{2.1}$$

$$p^\varepsilon_{\mathrm{app}}(t,x,y,z) = \sum_{i=0}^{n} \varepsilon^i \left(p^i(t,x,y,z) + \tilde{p}^i\left(t,x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{z}{\varepsilon}\right) + \bar{p}^i\left(t,x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{z-1}{\varepsilon}\right)\right); \tag{2.2}$$

- $u^i = u^i(t,x,y,z)$ (respectively $p^i = p^i(t,x,y,z)$) is an interior term, 1-periodic in (x,y), defined for $t \in \mathbb{R}^+$, and $(x,y,z) \in \Omega^\varepsilon$,
- $\tilde{u}^i = \tilde{u}^i(t,x,y,X,Y,Z)$ (respectively $\tilde{p}^i = \tilde{p}^i(t,x,y,X,Y,Z)$) is a lower boundary layer term, 1-periodic in (x,y) and (X,Y), defined for $t \in \mathbb{R}^+$, $(x,y) \in (0,1)^2$, $(X,Y,Z) \in \widetilde{\omega}$,
- $\bar{u}^i = \bar{u}^i(t,x,y,X,Y,Z)$ (respectively $\bar{p}^i = \bar{p}^i(t,x,y,X,Y,Z)$) is an upper boundary layer term, 1-periodic in (x,y) and (X,Y), defined for $t \in \mathbb{R}^+$, $(x,y) \in (0,1)^2$, $(X,Y,Z) \in \overline{\omega}$.

2.1.2. Boundary conditions

It remains to add boundary conditions on these profiles. We impose that interior terms equal zero outside the interior domain Ω: for all i,

$$\forall t \geqslant 0,\ \forall \mathbf{x} \in \Omega^\varepsilon - \Omega, \quad u^i(t,\mathbf{x}) = 0, \qquad p^i(t,\mathbf{x}) = 0. \tag{2.3}$$

The boundary layer terms satisfy:

$$\tilde{u}^i = 0 \quad \text{on } \tilde{\gamma}, \qquad \bar{u}^i = 0 \quad \text{on } \bar{\gamma}, \tag{2.4}$$

and are expected to play no role outside the layer, which can be written:

$$\tilde{u}^i, \bar{u}^i \xrightarrow[Z\to\pm\infty]{} 0, \qquad \tilde{p}^i, \bar{p}^i \xrightarrow[Z\to\pm\infty]{} 0. \tag{2.5}$$

We need further conditions at the interfaces. The reason is the following: in order to prove the convergence of u^ε to u^0, we need to carry energy estimates on $v^\varepsilon = u^\varepsilon - u^\varepsilon_{\mathrm{app}}$, $q^\varepsilon = p^\varepsilon - p^\varepsilon_{\mathrm{app}}$. In these estimates, treatment of pressure and viscosity terms leads to:

$$\begin{aligned}&\int \varepsilon^{-1}\nabla q^\varepsilon \cdot v^\varepsilon - \varepsilon\int \Delta v^\varepsilon \cdot v^\varepsilon\\ &\quad = +\int |\nabla v^\varepsilon|^2 + \int_{\Sigma_1\cup\Sigma_2} \left[\left(\varepsilon\frac{\partial u^\varepsilon_{\mathrm{app}}}{\partial \mathbf{n}} - \varepsilon^{-1} p^\varepsilon_{\mathrm{app}}\vec{n}\right)\cdot u^\varepsilon_{\mathrm{app}}\right]\Big|_{\Sigma_1\cup\Sigma_2}.\end{aligned}$$

We see that this last surface integral must be small enough, so that energy estimates allow to conclude. Note that it might not be the case if $u^\varepsilon_{\mathrm{app}}$ was only including interior terms, because interior terms have a priori strong discontinuities at $\Sigma_1 \cup \Sigma_2$ (and boundary layer terms are added to compensate these discontinuities). Sufficient conditions for this integral to vanish are the following jump conditions: $\forall (x, y) \in \mathbb{T}^2$,

$$\left[\tilde{u}^i(t,x,y,\cdot)\right]\big|_\Sigma = -\left[u^i(t,x,y,\cdot)\right]\big|_{z=0} \tag{2.6}$$

(which expresses a natural continuity condition of $u^\varepsilon_{\mathrm{app}}$ at the interface Σ_1),

$$\left[\tilde{p}^0(t,x,y,\cdot)\right]\big|_\Sigma = -\left[p^0(t,x,y,\cdot)\right]\big|_{z=0}, \tag{2.7}$$

$$\left[\left(\frac{\partial \tilde{u}^0}{\partial Z} - \tilde{p}^1\mathbf{e}\right)(t,x,y,\cdot)\right]\Big|_\Sigma = -\left[-p^1(t,x,y,\cdot)\mathbf{e}\right]\big|_{z=0}, \tag{2.8}$$

$$\left[\left(\frac{\partial \tilde{u}^i}{\partial Z} - \tilde{p}^{i+1}\mathbf{e}\right)(t,x,y,\cdot)\right]\Big|_\Sigma = -\left[\left(\frac{\partial u^{i-1}}{\partial z} - p^{i+1}\right)(t,x,y,\cdot)\mathbf{e}\right]\Big|_{z=0}, \quad i \geqslant 1. \tag{2.9}$$

Similar jump conditions hold of course for the upper boundary layer.

Remark. Let us motivate the use of such interfaces and jump conditions. Indeed, in the flat case $\mathbb{T}^2 \times (0,1)$, Grenier and Masmoudi [13] used a simpler expansion, namely:

$$u^\varepsilon_{\mathrm{app}} = \sum \varepsilon^i\left(u^i_{\mathrm{int}}(t,x,y,z) + u^i_b\left(x,y,\varepsilon^{-1}z\right) + u^i_t\left(x,y,\varepsilon^{-1}(1-z)\right)\right)$$

and Dirichlet condition on $u^\varepsilon_{\mathrm{app}}$ was expressed through the following natural equalities,

$$u^i_b(x,y,0) = -u^i_{\mathrm{int}}(x,y,0), \qquad u^i_t(x,y,0) = -u^i_{\mathrm{int}}(x,y,1). \tag{2.10}$$

In the same spirit, the apparent natural choice here was to consider expansion:

$$\begin{aligned}u^\varepsilon_{\mathrm{app}} = \sum \varepsilon^i\big(&u^i_{\mathrm{int}}(t,x,y,z) + u^i_b\left(x,y,\varepsilon^{-1}x,\varepsilon^{-1}y,\varepsilon^{-1}z\right)\\ &+ u^i_t\left(x,y,\varepsilon^{-1}x,\varepsilon^{-1}y,\varepsilon^{-1}(z-1)\right)\big),\end{aligned}$$

and express the Dirichlet condition using the values of the profiles at the boundary. However, if we chose for instance a parametrization $\gamma = \gamma(\varepsilon^{-1}x, \varepsilon^{-1}y)$ of the bottom boundary, the Dirichlet condition leads to:

$$\sum_i \varepsilon^i u_{\text{int}}^i\big(x, y, \varepsilon\gamma(X, Y)\big) = -\sum_i \varepsilon^i u_b^i\big(x, y, X, Y, \gamma(X, Y)\big)$$

which through a Taylor expansion in the variable z, becomes:

$$\sum_i \varepsilon^i \left(\sum_j \big(\varepsilon\gamma(X, Y)\big)^j \partial_z^j u_{\text{int}}^j(x, y, 0)\right) = -\sum_i \varepsilon^i \left(u_b^i\big(x, y, X, Y, \gamma(X, Y)\big)\right). \tag{2.11}$$

Thus, relations equivalent to (2.10) involve derivatives of the interior terms at arbitrary order. The formulation with interfaces and jump conditions involves only first order derivatives, making the mathematical study more tractable. Higher order derivatives are “hidden within the pressure term”.

2.2. *Equations on the profiles*

We plug Ansatz (2.1) and (2.2) in Eqs. (0.3), (0.4). The resulting equations are ordered according to powers of ε, and coefficients of the different powers of ε are set equal to zero. It leads to a collection of equations on the profiles. We will focus on the lower boundary layer, the upper one leading to similar equation.

At order ε^{-2} in the boundary layer, we get ($\mathbf{X} = (X, Y, Z)$)

$$\nabla_{\mathbf{X}} \tilde{p}^0 = 0 \quad \text{in } \widetilde{\omega}^- \cup \widetilde{\omega}^+. \tag{2.12}$$

The pressure does not change in the boundary layer, which is classical (see [18]). To satisfy jump condition (2.7), we shall take:

$$\begin{aligned} &\tilde{p}^0 = 0 \quad \text{in } \widetilde{w}^+, \\ &\tilde{p}^0 = -\big[p^0\big]\big|_{z=0} \quad \text{in } \widetilde{w}^-. \end{aligned} \tag{2.13}$$

At order ε^{-1} in the interior, Eq. (0.3) yields

$$\mathbf{e} \times u^0 + \nabla p^0 = 0. \tag{2.14}$$

At order ε^0 in the interior, we get from (0.4)

$$\nabla \cdot u^0 = 0. \tag{2.15}$$

Using the last line of (2.14), we have:

$$\partial_z p^0 = 0 \tag{2.16}$$

and taking the curl of (2.14) together with (2.16), we obtain:

$$\partial_z u^0 = 0. \tag{2.17}$$

Thus, p^0 and u^0 are independent of z. This is the well-known Taylor–Proudman Theorem.

At order ε^{-1} in the boundary layer, we get from (0.4)

$$\nabla_{\mathbf{X}} \cdot \tilde{u}^0 = 0 \quad \text{in } \widetilde{\omega}^- \cup \widetilde{\omega}^+. \tag{2.18}$$

Applying Green–Ostrogradsky formula with (2.4), (2.5),

$$0 = \int_{\widetilde{\omega}} \nabla_{\mathbf{X}} \cdot \tilde{u}^0 = \left[\tilde{u}_3^0\right]\big|_{\Sigma}.$$

Using jump condition (2.6), and the fact that u_3^0 is independent of z, we then have:

$$u_3^0 = 0. \tag{2.19}$$

At order ε^{-1} in the boundary layer, we get from (0.3):

$$\mathbf{e} \times \tilde{u}^0 + \tilde{u}^0 \cdot \nabla_{\mathbf{X}} \tilde{u}^0 + \nabla \tilde{p}^0 + \nabla_{\mathbf{X}} \tilde{p}^1 - \Delta_{\mathbf{X}} \tilde{u}^0 = 0 \tag{2.20}$$

which using (2.13), (2.14) can be written:

$$\mathbf{e} \times \tilde{u}^0 + \tilde{u}^0 \cdot \nabla_{\mathbf{X}} \tilde{u}^0 + \nabla_{\mathbf{X}} \tilde{p}^1 - \Delta_{\mathbf{X}} \tilde{u}^0 = -\begin{pmatrix} u^{0,\perp} \\ 0 \end{pmatrix}, \quad \text{in } \widetilde{\omega}^-, \tag{2.21}$$

$$\mathbf{e} \times \tilde{u}^0 + \tilde{u}^0 \cdot \nabla_{\mathbf{X}} \tilde{u}^0 + \nabla_{\mathbf{X}} \tilde{p}^1 - \Delta_{\mathbf{X}} \tilde{u}^0 = 0, \quad \text{in } \widetilde{\omega}^+. \tag{2.22}$$

Jump and boundary conditions can be written:

$$\left[\tilde{u}^0(t, x, y, \cdot)\right]\big|_{\Sigma} = -u^0(t, x, y), \tag{2.23}$$

$$\left[\left(\frac{\partial \tilde{u}^0}{\partial Z} - \tilde{p}^1\right)(t, x, y, \cdot)\right]\bigg|_{\Sigma} = \left[p^1(t, x, y, \cdot)\right]\big|_{z=0}, \tag{2.24}$$

$$\tilde{u}^0 = 0 \quad \text{on } \tilde{\gamma}. \tag{2.25}$$

Up to modify $\tilde{p}^1$ in $\widetilde{\omega}^-$ by:

$$\tilde{p}^1 := \tilde{p}^1 + \left[p^1(t, x, y, \cdot)\right]\big|_{z=0}$$

which does not change Eq. (2.20), we may suppose that

$$\left[\left(\frac{\partial \tilde{u}^0}{\partial Z} - \tilde{p}^1\right)(t, x, y, \cdot)\right]\bigg|_{\Sigma} = 0. \tag{2.26}$$

Thus, if we gather (2.18), (2.21), (2.22), (2.23), (2.26), (2.24), we see that $\tilde{u}^0$ solves a system of type (BL) with t, x, y being simply parameters.

Let us now identify the system satisfied by $u^0 = \binom{u(t,x,y)}{0}$. At order ε^0, we have from (0.3):

$$\partial_t u^0 + u^0 \cdot \nabla u^0 + \mathbf{e} \times u^1 + \nabla p^1 = 0 \tag{2.27}$$

and at order ε^1 from (0.4),

$$\nabla \cdot u^1 = 0. \tag{2.28}$$

As shown in [18], noting $\zeta^0 = \partial_x u_2^0 - \partial_y u_1^0$ the curl of u^0, we get:

$$\partial_t \zeta^0 + u^0 \cdot \nabla \zeta^0 = \partial_z u_3^1.$$

Then, we integrate for z from 0 to 1. As u^0, ζ^0 are independent of z, we get:

$$\begin{aligned}\partial_t \zeta^0 + u^0 \cdot \nabla \zeta^0 &= u_3^1(\cdot\,, z=1) - u_3^1(\cdot\,, z=0) = -\big[u_3^1\big]\big|_{z=1} - \big[u_3^1\big]\big|_{z=0}\\ &= \big[\bar{u}_3^1\big]\big|_{\Sigma} + \big[\tilde{u}_3^1\big]\big|_{\Sigma}.\end{aligned}$$

Computation of $[\tilde{u}_3^1]|_{\Sigma}$. Eq. (0.4) gives, at order ε^0 in the boundary layer,

$$\nabla_{\mathbf{X}} \cdot \tilde{u}^1 + \partial_x \tilde{u}_1^0 + \partial_y \tilde{u}_2^0 = 0.$$

Applying again Green–Ostrogradsky formula:

$$\big[\tilde{u}_3^1\big]\big|_{\Sigma} = \int_{\widetilde{\omega}} \big(\partial_x \tilde{u}_1^0 + \partial_y \tilde{u}_2^0\big)\,\mathrm{d}\mathbf{X}$$

and in the same way

$$\big[\bar{u}_3^1\big]\big|_{\Sigma} = \int_{\overline{\omega}} \big(\partial_x \bar{u}_1^0 + \partial_y \bar{u}_2^0\big)\,\mathrm{d}\mathbf{X},$$

so that ζ^0 solves

$$\partial_t \zeta^0 + u^0 \cdot \nabla \zeta^0 + \mathrm{curl}\left(\int_{\widetilde{\omega}} \begin{pmatrix} \tilde{u}_2^0 \\ -\tilde{u}_1^0 \end{pmatrix} \mathrm{d}\mathbf{X} + \int_{\widetilde{\omega}} \begin{pmatrix} \bar{u}_2^0 \\ -\bar{u}_1^0 \end{pmatrix} \mathrm{d}\mathbf{X}\right) = 0.$$

Let us define the operator $\widetilde{P}$ for $\mathbf{u} \in \mathbb{R}^2$, by:

$$\widetilde{P}(\mathbf{u}) = \int_{\widetilde{\omega}} \begin{pmatrix} \tilde{u}_2 \\ -\tilde{u}_1 \end{pmatrix},$$

where $\tilde{u}$ is solution of (BL). We define in the same way function $\overline{P}$ for the upper layer. Note that the definition of $\widetilde{P}$ and $\overline{P}$ is still formal at this level: a precise meaning will be given to them in Section 4. Setting $P = \widetilde{P} + \overline{P}$, we get that u is (formally) solution of (Int).

Through this formal expansion, we have identified the equations necessarily satisfied by the interior and layer profiles. In the sections to come, we will show the well-posedness of these auxiliary problems, and then show that the approximate solutions we have constructed are close to exact solutions of Navier–Stokes–Coriolis equations.

3. The boundary layer

The aim of this section is to prove Theorem 1.1.

3.1. *Resolution of (BL)*

We prove the existence of a solution $\tilde{u}$ to (BL). Let $\mathbf{u} \in \mathbb{R}^2$. Let U_E be the "Ekman flow" defined by:

$$\begin{aligned}
&U_{E,1}(X,Y,Z) = -\mathrm{e}^{-Z\sqrt{2}}\left(u_1 \cos\left(\frac{Z}{\sqrt{2}}\right) + u_2 \sin\left(\frac{Z}{\sqrt{2}}\right)\right) \quad \text{in } \widetilde{\omega}^+,\\
&U_{E,2}(X,Y,Z) = -\mathrm{e}^{-Z\sqrt{2}}\left(u_2 \cos\left(\frac{Z}{\sqrt{2}}\right) - u_1 \sin\left(\frac{Z}{\sqrt{2}}\right)\right) \quad \text{in } \widetilde{\omega}^+,\\
&U_{E,3}(X,Y,Z) = 0 \quad \text{in } \widetilde{\omega}^+, \qquad U_E = 0 \quad \text{in } \widetilde{\omega}^-
\end{aligned} \tag{3.1}$$

and let $\Pi_E = 0$. It is known from [13,18] that (U_E, Π_E) solves the second equation of (BL). Setting $\tilde{u} = U_E + v$, the solvability of (BL) is equivalent to the solvability of:

$$\begin{aligned}
&\mathbf{e} \times v + \nabla \tilde{p} + U_E \cdot \nabla v + v \cdot \nabla U_E + v \cdot \nabla v - \Delta v = \begin{pmatrix} -\mathbf{u}^\perp \\ 0 \end{pmatrix} \quad \text{in } \widetilde{\omega}^-,\\
&\mathbf{e} \times v + \nabla \tilde{p} + U_E \cdot \nabla v + v \cdot \nabla U_E + v \cdot \nabla v - \Delta v = 0 \quad \text{in } \widetilde{\omega}^+,\\
&\nabla \cdot v = 0 \quad \text{in } \widetilde{\omega}^+ \cup \widetilde{\omega}^-,\\
&[v]|_\Sigma = 0 \quad \text{at } \Sigma,\\
&\left[\frac{\partial v}{\partial Z} - \tilde{p}\mathbf{e}\right]\bigg|_\Sigma = -\left[\frac{\partial U_E}{\partial Z}\right]\bigg|_\Sigma \quad \text{on } \Sigma,\\
&v = 0 \quad \text{on } \tilde{\gamma},\ v \text{ 1-periodic in } (X,Y).
\end{aligned} \tag{BL2}$$

3.1.1. *Functional setting*

Let $\widetilde{\Gamma} = \bigcup_{k\in\mathbb{Z}^2}(\tilde{\gamma} + (k,0))$, $\widetilde{\Omega} = \bigcup_{k\in\mathbb{Z}^2}(\widetilde{\omega} + (k,0))$. Let $\mathcal{V} = \{\varphi \in \mathcal{C}^\infty(\widetilde{\Omega})^3$, 1-periodic in (X,Y), $\nabla \cdot \varphi = 0$, $\operatorname{supp} \varphi \cap \tilde{\gamma} = \emptyset\}$. Let V the adherence of $\mathcal{V}$ in

$$\left\{u \in L^2_{\mathrm{loc}}(\widetilde{\omega})^3,\ \nabla u \in L^2(\widetilde{\omega})^9,\ u \in L^2(\widetilde{\omega}^-)^3,\ u = 0 \text{ on } \tilde{\gamma}\right\},$$

for the norm $\|v\|_V = (\int_{\widetilde{\omega}} |\nabla v|^2)^{1/2}$. V is a Hilbert space. We need to have a special basis of V. For this, for all $k > 0$, let

$$V^k = \{v \in V,\ \operatorname{supp} v \subset \{z < k\}\}.$$

It is straightforward to show the following:

Lemma 3.1. *There exists an orthonormal basis $\{\psi_n,\ n \in \mathbb{N}\}$ of V, such that for all k, there exists $I_k \subset \mathbb{N}$, $\{\psi_n,\ n \in I_k\}$ is a basis of V^k.*

We now prove:

Theorem 3.2. *There exists U_∞ such that for $|\mathbf{u}| \leqslant U_\infty$,* (BL2) *has a weak solution*: *more precisely, there exists $v \in V$ such that*: $\forall \varphi \in \mathcal{V}$,

$$\int_{\widetilde{\omega}} (\mathbf{e} \times v) \cdot \varphi + \int_{\widetilde{\omega}} (U_E \cdot \nabla v) \cdot \varphi + \int_{\widetilde{\omega}} (v \cdot \nabla U_E) \cdot \varphi + \int_{\widetilde{\omega}} (v \cdot \nabla v) \cdot \varphi + \int_{\widetilde{\omega}} \nabla v \cdot \nabla \varphi$$
$$= \begin{pmatrix} -\mathbf{u}^\perp \\ 0 \end{pmatrix} \cdot \int_{\widetilde{\omega}^-} \varphi + \left[\frac{\partial U_E}{\partial Z}\right]\Big|_\Sigma \cdot \int_\Sigma \varphi. \tag{3.2}$$

Proof. Following Temam [19] or Galdi [11], we use a Galerkin scheme. Let (ψ_n) the basis given by Lemma 3.1. We consider the sequence of approximate problems:

(BL^n): Find $v^n = \sum_{k=0}^n \alpha_k \psi_k$ such that for all k in $\{0, \ldots, n\}$,

$$\int_{\widetilde{\omega}} (\mathbf{e} \times v^n) \cdot \psi_k + \int_{\widetilde{\omega}} (U_E \cdot \nabla v^n) \cdot \psi_k + \int_{\widetilde{\omega}} (v^n \cdot \nabla U_E) \cdot \psi_k$$
$$+ \int_{\widetilde{\omega}} (v^n \cdot \nabla v^n) \cdot \psi_k + \int_{\widetilde{\omega}} \nabla v^n \cdot \nabla \psi_k = \begin{pmatrix} -\mathbf{u}^\perp \\ 0 \end{pmatrix} \cdot \int_{\widetilde{\omega}^-} \psi_k + \left[\frac{\partial U_E}{\partial Z}\right]\Big|_\Sigma \cdot \int_\Sigma \psi_k. \tag{3.3}$$

3.1.2. Resolution of (*BL^n*)

We start with an a priori estimate. Multiplying the last equation by α_k and summing over k leads to:

$$\int |\nabla v^n|^2 = -\int \big((U_E + v^n) \cdot \nabla v^n\big) \cdot v^n - \int (v^n \cdot \nabla U_E) \cdot v^n$$
$$+ \begin{pmatrix} -\mathbf{u}^\perp \\ 0 \end{pmatrix} \cdot \int_{\widetilde{\omega}^-} v^n + \left[\frac{\partial U_E}{\partial Z}\right]\Big|_\Sigma \cdot \int_\Sigma v^n$$
$$= I_1 + I_2 + I_3 + I_4.$$

We now bound every term of the right-hand side:

- It is well known that $I_1 = 0$.
- Integrals I_3 and I_4 satisfy, respectively:

$$
\begin{aligned}
|I_3| &\leqslant C\,|\mathbf{u}|\|\nabla v^n\|_{L^2(\widetilde{\omega}^-)},\\
|I_4| &\leqslant C_1|\mathbf{u}|\|v^n\|_{L^2(\Sigma)} \leqslant C_2|\mathbf{u}|\|v^n\|_{H^1(\widetilde{\omega}^-)} \leqslant C_3|\mathbf{u}|\|\nabla v^n\|_{L^2(\widetilde{\omega}^-)}.
\end{aligned}
$$

Here, we have used both the continuity of the trace function,

$$\|v\|_{H^{1/2}(\partial\mathcal{O})} \leqslant \|v\|_{H^1(\mathcal{O})}$$

and Poincaré's inequality:

$$\|v\|_{L^2(\mathcal{O})} \leqslant \|\nabla v\|_{L^2(\mathcal{O})},$$

which is valid for all functions cancelling on a non-zero measure set of $\partial\mathcal{O}$.

- Integral I_2:

$$
\begin{aligned}
I_2 &\leqslant \sum_{i=1}^{2}\left|\int_{\widetilde{\omega}^+} v_3^n \frac{\partial U_{E,i}}{\partial Z} v_i^n\right|\\
&\leqslant \sum_{i=1}^{2}\left|\int_{(0,1)^2} \mathrm{d}X\,\mathrm{d}Y \int_0^{+\infty} \mathrm{d}Z\left(\int_0^Z \frac{\partial v_3^n}{\partial\theta}\,\mathrm{d}\theta + v_3^n(X,Y,0)\right)\left(\frac{\partial U_{E,i}}{\partial Z}\right)\right.\\
&\qquad\left.\times\left(\int_0^Z \frac{\partial v_i^n}{\partial\theta}\,\mathrm{d}\theta + v_i^n(X,Y,0)\right)\right|,
\end{aligned}
$$

$$
\begin{aligned}
I_2 \leqslant{}& 2\int_{(0,1)^2} \mathrm{d}X\,\mathrm{d}Y \int_0^{+\infty} \mathrm{d}Z\,|Z|\left(\int_0^Z \left|\frac{\partial v^n}{\partial\theta}\right|^2 \mathrm{d}\theta\right)\left|\frac{\partial U_E}{\partial Z}\right|\\
&+ 4\int_{(0,1)^2} \mathrm{d}X\,\mathrm{d}Y \int_0^{+\infty} \mathrm{d}Z\,|Z|^{1/2}\left(\int_0^z \left|\frac{\partial v^n}{\partial\theta}\right|^2 \mathrm{d}\theta\right)^{1/2} \left|v^n(X,Y,0)\right|^2\left|\frac{\partial U_E}{\partial Z}\right|\\
&+ 2\int_{(0,1)^2} \mathrm{d}X\,\mathrm{d}Y\left|v^n(X,Y,0)\right|^2 \int_0^{+\infty} \mathrm{d}Z\left|\frac{\partial U_E}{\partial Z}\right|,
\end{aligned}
$$

and thus

$$\begin{aligned}
I_2 &\leqslant C_1 \|\nabla v^n\|^2_{L^2(\widetilde{\omega}^+)} \int_0^{+\infty} \mathrm{d}Z |Z| \left|\frac{\partial U_E}{\partial Z}\right| \\
&\quad + C_2 \|\nabla v^n\|_{L^2(\widetilde{\omega}^-)} \|\nabla v^n\|_{L^2(\widetilde{\omega}^+)} \int_0^{+\infty} \mathrm{d}Z |Z|^{1/2} \left|\frac{\partial U_E}{\partial Z}\right| \\
&\quad + C_3 \|\nabla v^n\|^2_{L^2(\widetilde{\omega}^-)} \int_0^{+\infty} \mathrm{d}Z \left|\frac{\partial U_E}{\partial Z}\right| \\
&\leqslant C \left(\int_0^{+\infty} \mathrm{d}Z \big(1 + |Z|^{1/2} + |Z|\big) \left|\frac{\partial U_E}{\partial Z}\right| \right) \|\nabla v^n\|^2_{L^2(\widetilde{\omega})}.
\end{aligned}$$

Gathering all these bounds, we get:

$$\|\nabla v^n\|^2_{L^2} \leqslant C\big(|\mathbf{u}|\, \|\nabla v^n\|_{L^2} + |\mathbf{u}|\, \|\nabla v^n\|^2_{L^2}\big).$$

For $|u|$ small enough, we obtain:

$$\|\nabla v^n\|_{L^2} \leqslant C' |\mathbf{u}|. \tag{3.4}$$

On the basis of estimate (3.4), it is then standard to conclude for the existence of a solution v^n to (BL^n). For instance, define $V_n = \mathrm{span}\{\psi_0, \ldots, \psi_n\}$ and for all w in V_n, set $F(w) =$ the solution w^n in V_n of: for all ψ in V_n,

$$\begin{aligned}
&\int_{\widetilde{\omega}} (\mathbf{e} \times w^n) \cdot \psi + \int_{\widetilde{\omega}} (U_E \cdot \nabla w^n) \cdot \psi + \int_{\widetilde{\omega}} (w^n \cdot \nabla U_E) \cdot \psi \\
&\quad + \int_{\widetilde{\omega}} (w \cdot \nabla w^n) \cdot \psi + \int_{\widetilde{\omega}} \nabla w^n \cdot \nabla \psi = \begin{pmatrix} -\mathbf{u}^\perp \\ 0 \end{pmatrix} \cdot \int_{\widetilde{\omega}^-} \psi + \left[\frac{\partial U_E}{\partial Z}\right]\Big|_{\Sigma} \cdot \int_{\Sigma} \psi.
\end{aligned} \tag{3.5}$$

Proceeding as for inequality (3.4), one shows that $\|F(w)\|_V \leqslant C|\mathbf{u}|$. In particular,

$$\big\langle F(w) - w, w \big\rangle_V < 0, \quad \text{for } R \text{ large enough and } \|w\| = R.$$

One concludes that F has a fixed point, thanks to Brouwer theorem. For more details, we refer to [11, Lemma 3.2, p. 30].

3.1.3. Convergence of v^n

Because of the previous estimate, if $|\mathbf{u}|$ is small enough, (v^n) is bounded in V, and converges weakly to $v \in V$. Moreover, for all compact $K \subset \widetilde{\Omega}$, (v^n) converges strongly to v in $L^2(K)$, up to a subsequence (this is due to Rellich's Theorem, see [3]). Using that for all k, $K = \mathrm{Supp}(\psi_k)$ is compact, we may pass to the limit and find: for all k,

$$\int_{\widetilde{\omega}} (\mathbf{e} \times v) \cdot \psi_k + \int_{\widetilde{\omega}} (U_E \cdot \nabla v) \cdot \psi_k + \int_{\widetilde{\omega}} (v \cdot \nabla U_E) \cdot \psi_k + \int_{\widetilde{\omega}} (v \cdot \nabla v) \cdot \psi_k + \int_{\widetilde{\omega}} \nabla v \cdot \nabla \psi_k$$
$$= \begin{pmatrix} -\mathbf{u}^{\perp} \\ 0 \end{pmatrix} \cdot \int_{\widetilde{\omega}^-} \psi_k + \left[\frac{\partial U_E}{\partial Z}\right]\Big|_{\Sigma} \cdot \int_{\Sigma} \psi_k. \tag{3.6}$$

Let now $\varphi \in \mathcal{V}$. There exists k, such that $\varphi \in V^k$. Using the density of $\mathrm{span}\{\psi_n,\ n \in I_k\}$ in V^k (cf. Lemma 3.1), it is still easy to pass on the limit as $n \to \infty$ to obtain:

$$\int_{\widetilde{\omega}} (\mathbf{e} \times v) \cdot \varphi + \int_{\widetilde{\omega}} (U_E \cdot \nabla v) \cdot \varphi + \int_{\widetilde{\omega}} (v \cdot \nabla U_E) \cdot \varphi + \int_{\widetilde{\omega}} (v \cdot \nabla v) \cdot \varphi + \int_{\widetilde{\omega}} \nabla v \cdot \nabla \varphi$$
$$= \begin{pmatrix} -\mathbf{u}^{\perp} \\ 0 \end{pmatrix} \cdot \int_{\widetilde{\omega}^-} \varphi \left[\frac{\partial U_E}{\partial Z}\right]\Big|_{\Sigma} \cdot \int_{\Sigma} \varphi \tag{3.7}$$

which ends the proof of the theorem. □

Remarks. (1) We could have expected in the variational formulation (3.2) to replace $\varphi \in \mathcal{V}$ by $\varphi \in \widetilde{V}$, where $\widetilde{V}$ is the adherence of $\mathcal{V}$ for the norm $\|u\|_{\widetilde{V}} = \|\nabla u\|_{L^2} + \|u\|_{L^3}$, as it is the case with Dirichlet boundary conditions (see [19]). This is not possible, because we do not know a priori if v belongs to $H^1(\widetilde{\omega})$, and so we cannot use the Sobolev injection:

$$H^1(\mathbb{R}^n) \hookrightarrow L^{2n/(n-2)}(\mathbb{R}^n), \quad n \geqslant 3.$$

Therefore, $\int (v \cdot \nabla v) \cdot \varphi$ is not a priori defined if the support of φ is not compact. For the same reason, we needed to remain in a fixed compact to pass to the limit in the quadratic terms above.

Moreover, it shows that, if we wish to use Sobolev injections (as it is the case in the next part), we need to distinguish between the oscillatory part of v (for which injections hold) and its average.

(2) By De Rham's Lemma, to each variational solution $\tilde{u}$, we can associate a function $\tilde{p} \in L^2_{\mathrm{loc}}(\widetilde{\Omega})$, unique up to an additive constant, such that $\tilde{u}$, $\tilde{p}$ satisfies the first two lines of (BL) in the distribution sense. Moreover, by classical elliptic regularity results,

$$\tilde{u}, \tilde{p} \in \mathcal{C}^{\infty}\big(\widetilde{\Omega} \cap \{Z > 0\} \cup \widetilde{\Omega} \cap \{Z < 0\}\big).$$

3.2. De Saint-Venant estimates

The aim of this section is to prove that the solutions of (BL) are real boundary layer terms, i.e., that they satisfy the so-called "De Saint-Venant estimate" given in Theorem 1.1. To do this, we will follow ideas of [10,11] relative to steady flows in semi-infinite straight channels. Let $\tilde{u}$, $\tilde{p}$ a solution of (BL). That means $\tilde{u} = U_E + v$, with U_E given by (3.1), and v given by Theorem 3.2. For the sake of simplicity, we drop the tildas on $\tilde{u}$, $\tilde{p}$ within this section, and set $u := \tilde{u}$, $p := \tilde{p}$. Note that through standard regularity results (see last remark of previous section), (u, p) is smooth on $\widetilde{\omega}^+$ and therefore is a classical solution of the second equation of (BL).

The proof is divided into three steps:

1. We distinguish horizontal average $\langle u \rangle$ from oscillatory part u^* of solution u.
2. Using elliptic regularity results, we show that it is enough to get the exponential decay of $f(R) = \|\nabla u^*\|_{L^2(\widetilde{\omega}\cap\{z>R\})}$.
3. Finally, we obtain the exponential control of $f(R)$ as R goes to infinity, thanks to a Gronwall type inequality on f.

As already mentioned, we focus on the lower boundary layer, but similar results hold for the upper one.

3.2.1. Notations

We need to define a few notations which will be used in the sequel. First, we remind notations:

$$\forall R \geqslant 0, \quad \widetilde{\omega}^R = \widetilde{\omega} \cap \{Z > R\},$$

$$\forall R_1, R_2 \geqslant 0, \quad \widetilde{\omega}^{R_1,R_2} = \widetilde{\omega} \cap \{R_2 > Z > R_1\}.$$

Then, to all $w \in L^1_{\mathrm{loc}}(\mathbb{T}^2 \times \mathbb{R}^+)^N$ $(N \geqslant 1)$, we associate its average $\langle w \rangle \in L^1_{\mathrm{loc}}(\mathbb{R}^+)^N$ and its oscillatory part $w^* \in L^1_{\mathrm{loc}}(\mathbb{T}^2 \times \mathbb{R}^+)^N$, namely,

$$\forall \mathbf{X} = (X, Y, Z) \in \mathbb{T}^2 \times \mathbb{R}^+, \quad \langle w \rangle(Z) = \int_{\Sigma(Z)} w \,\mathrm{d}X\,\mathrm{d}Y, \quad w^*(\mathbf{X}) = w(\mathbf{X}) - \langle w \rangle(Z),$$

where, for all $R \geqslant 0$, $\Sigma(R)$ is the cross-section at $Z = R$. In order to lighten notations, we may sometimes consider $\langle w \rangle$ as a function of $\mathbf{X} \in \widetilde{\omega}^+$ instead of $Z > 0$.

Finally, we introduce notation $\|\cdot\|_{m,q,R_1,R_2}$, to refer to the norm of $W^{m,q}(\widetilde{\omega}^{R_1,R_2})^3$.

3.2.2. Average and oscillations

Lemma 3.3. $u^* \in H^1(\widetilde{\omega}^+)^3$, $\langle u_3 \rangle = 0$.

Proof. Clearly,

$$\nabla u \in L^2(\widetilde{\omega}^+)^9 \ \Rightarrow\ u^* \in H^1(\widetilde{\omega}^+)^3.$$

As $[u_3]|_\Sigma = 0$, and $\nabla \cdot u = 0$, we have:

$$0 = \int_{\widetilde{\omega}^-} \nabla \cdot u = \int_\Sigma u_3 \,\mathrm{d}X \,\mathrm{d}Y,$$
$$0 = \int_{\widetilde{\omega}^{0,Z}} \nabla \cdot u = -\int_\Sigma u_3 \,\mathrm{d}X \,\mathrm{d}Y + \langle u_3 \rangle(Z)$$

so that $\langle u_3 \rangle = 0$. $\square$

Functions $\langle u \rangle$, $\langle p \rangle$, u^*, p^* satisfy in $\widetilde{\omega}^+$,

$$\mathbf{e} \times \langle u \rangle + \begin{pmatrix} 0 \\ 0 \\ \partial_Z \end{pmatrix} \langle p \rangle - \partial_Z^2 \langle u \rangle + \langle u \cdot \nabla u \rangle = 0, \tag{3.8}$$
$$\mathbf{e} \times u^* + \nabla p^* - \Delta u^* + (u \cdot \nabla u)^* = 0, \tag{3.9}$$
$$\nabla \cdot u^* = 0. \tag{3.10}$$

Using that $\langle u_3 \rangle = 0$, we have:

$$\langle u \cdot \nabla u \rangle = \langle u^* \cdot \nabla u^* \rangle, \quad (u \cdot \nabla u)^* = (u^* \cdot \nabla u^*)^*.$$

Thus, Eqs. (3.8), (3.9), (3.10) turn into:

$$-\langle u_2 \rangle - \frac{\partial^2 \langle u_1 \rangle}{\partial Z^2} = \big\langle (u^* \cdot \nabla u^*)_1 \big\rangle, \tag{3.11}$$
$$+\langle u_1 \rangle - \frac{\partial^2 \langle u_2 \rangle}{\partial Z^2} = \big\langle (u^* \cdot \nabla u^*)_2 \big\rangle, \tag{3.12}$$
$$\frac{\partial p}{\partial Z} = \big\langle (u^* \cdot \nabla u^*)_3 \big\rangle, \tag{3.13}$$
$$\mathbf{e} \times u^* + \nabla p^* - \Delta u^* + (u^* \cdot \nabla u^*)^* = 0, \tag{3.14}$$
$$\nabla \cdot u^* = 0. \tag{3.15}$$

System (3.11), (3.12) is a simple differential system. This will allow us to have precise information on $\langle u \rangle$ relatively to the right member $\langle (u^* \cdot \nabla u^*) \rangle$.

Let us define:

$$V = \langle u_1 \rangle + \mathrm{i}\langle u_2 \rangle, \qquad F = \big\langle (u^* \cdot \nabla u^*)_1 \big\rangle + \mathrm{i}\big\langle (u^* \cdot \nabla u^*)_2 \big\rangle, \quad \mathrm{i}^2 = -1.$$

Eqs. (3.11), (3.12) are equivalent to:

$$\mathrm{i}V - V'' = F. \tag{3.16}$$

This differential equation can be written as a first-order system:

$$\begin{pmatrix} V \\ V' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ \mathrm{i} & 0 \end{pmatrix} \begin{pmatrix} V \\ V' \end{pmatrix} + \begin{pmatrix} 0 \\ -F \end{pmatrix} = \mathbf{A} \begin{pmatrix} V \\ V' \end{pmatrix} + \begin{pmatrix} 0 \\ -F \end{pmatrix}.$$

$\mathbf{A}$ is diagonalizable, with eigenvalues $(1+\mathrm{i})/\sqrt{2}$, $-(1+\mathrm{i})/\sqrt{2}$. Let P^+ and P^- the associated eigenprojections.

$\left(\begin{smallmatrix} V \\ V' \end{smallmatrix}\right)$ can be written, for a $\left(\begin{smallmatrix} V_0 \\ W_0 \end{smallmatrix}\right) \in \mathbb{C}^2$, for all $Z > 0$,

$$\begin{aligned} \begin{pmatrix} V \\ V' \end{pmatrix}(Z) = \mathrm{e}^{\mathbf{A}Z} \begin{pmatrix} V_0 \\ W_0 \end{pmatrix} &+ \int_0^Z \mathrm{e}^{\mathbf{A}(Z-S)} P^- \begin{pmatrix} 0 \\ -F \end{pmatrix} \mathrm{d}S \\ &- \int_Z^{+\infty} \mathrm{e}^{\mathbf{A}(Z-S)} P^+ \begin{pmatrix} 0 \\ -F \end{pmatrix} \mathrm{d}S. \end{aligned} \tag{3.17}$$

Introducing the Green function of the problem:

$$\begin{cases} G(z) = \mathrm{e}^{\mathbf{A}Z} P^- & \text{if } Z > 0, \\ G(z) = \mathrm{e}^{-\mathbf{A}Z} P^+ & \text{if } Z < 0, \end{cases}$$

and

$$\begin{cases} f(z) = \begin{pmatrix} 0 \\ -F \end{pmatrix} & \text{if } Z > 0, \\ f(z) = 0 & \text{if } Z < 0, \end{cases}$$

we can write, for all $Z > 0$,

$$\begin{pmatrix} V \\ V' \end{pmatrix}(Z) = \mathrm{e}^{\mathbf{A}Z} \begin{pmatrix} V_0 \\ W_0 \end{pmatrix} + G * f. \tag{3.18}$$

Note that G satisfies, for positive constants C and α,

$$\forall Z, \quad \left|G(Z)\right| \leqslant C\mathrm{e}^{-\alpha|Z|},$$

and that $f \in L^1(\mathbb{R})$. More precisely, for all $R > 0$,

$$\begin{aligned} \int_{\{Z>R\}} |f| \leqslant \int_{\{Z>R\}} \left|\langle v^*.\nabla v^*\rangle\right| \mathrm{d}Z \leqslant \int_{\widetilde{\omega}^R} |v^*.\nabla v^*| \, \mathrm{d}\mathbf{X} \\ \leqslant \|v^*\|_{L^2(\widetilde{\omega}^R)} \|\nabla v^*\|_{L^2(\widetilde{\omega}^R)} \leqslant \|\nabla v^*\|^2_{L^2(\widetilde{\omega}^R)}. \end{aligned}$$

Let us then bound $\|G * f\|_{L^2(\{Z>R\})}$. We have:

$$\begin{aligned}\|G*f\|^2_{L^2(\{Z>R\})} &= \int\limits_{\{Z>R\}} \mathrm{d}Z\biggl(\int\limits_{\mathbb{R}} \mathrm{d}Z'\,G(Z-Z')f(Z')\biggr)^2\\ &\leqslant \int\limits_{\{Z>R\}} \mathrm{d}Z\biggl(\int\limits_{\mathbb{R}} \mathrm{d}Z'\bigl|f(Z')\bigr|\bigl|G(Z-Z')\bigr|^2\biggr)\biggl(\int\limits_{\mathbb{R}} \mathrm{d}Z'\bigl|f(Z')\bigr|\biggr)\\ &\leqslant M\int\limits_{\{Z>R\}} \mathrm{d}Z\biggl(\int\limits_{\mathbb{R}} \mathrm{d}Z'\bigl|f(Z')\bigr|\bigl|G(Z-Z')\bigr|^2\biggr),\end{aligned}\tag{3.19}$$

where we have used the Cauchy–Schwarz inequality, and where M is a constant depending on $\|\nabla u^*\|_{L^2(\omega)}$ (see the computation of $\|f\|_{L^1(\mathbb{R})}$). Now, we apply Fubini Theorem to last expression, and obtain:

$$\begin{aligned}&\int\limits_{\{Z>R\}} \mathrm{d}Z\biggl(\int\limits_{\mathbb{R}} \mathrm{d}Z'\bigl|f(Z')\bigr|\bigl|G(Z-Z')\bigr|^2\biggr)\\ &\quad\leqslant C\int\limits_{\mathbb{R}} \mathrm{d}Z'\bigl|f(Z')\bigr|\biggl(\int\limits_{\{Z>R\}} \mathrm{d}Z\,\mathrm{e}^{-2\alpha|Z-Z'|}\biggr).\end{aligned}\tag{3.20}$$

We deduce from this inequality:

$$\begin{aligned}\|G*f\|^2_{L^2(\{Z>R\})} &\leqslant \int\limits_{Z'<R/2} \mathrm{d}Z'\bigl|f(Z')\bigr|\biggl(\int\limits_{\{Z>R\}} \mathrm{d}Z\,\mathrm{e}^{-2\alpha(Z-Z')}\biggr)\\ &\quad+\int\limits_{Z'>R/2} \mathrm{d}Z'\bigl|f(Z')\bigr|\biggl(\int\limits_{\{Z>R\}} \mathrm{d}Z\,\mathrm{e}^{-2\alpha|Z-Z'|}\biggr) = I_1+I_2.\end{aligned}\tag{3.21}$$

We get:

$$I_1 = \frac{1}{2\alpha}\int\limits_{Z'<R/2} \mathrm{d}Z'\bigl|f(Z')\bigr|\mathrm{e}^{-2\alpha(R-Z')} \leqslant \frac{1}{2\alpha}\mathrm{e}^{-\alpha R}\|f\|_{L^1(\mathbb{R})},$$

and

$$I_2 \leqslant \int\limits_{Z'>R/2} \mathrm{d}Z'\bigl|f(Z')\bigr|\biggl(\int\limits_{\mathbb{R}} \mathrm{d}X\,\mathrm{e}^{-\alpha|X|}\biggr) \leqslant C\int\limits_{Z'>R/2} \mathrm{d}Z'\bigl|f(Z')\bigr|$$

so that using above computations on $\int_{Z>R}\mathrm{d}Z|f(Z)|$, we find $I_2 \leqslant C'\|\nabla u^*\|^2_{L^2(\widetilde{\omega}^{R/2})}$. Finally, for a $C>0$,

$$\|G*f\|^2_{L^2(\{Z>R\})} \leqslant C\bigl(\exp(-\alpha R) + \|\nabla u^*\|^2_{L^2(\widetilde{\omega}^{R/2})}\bigr).\tag{3.22}$$

As $V' \in L^2(\mathbb{R}^+)$ (cf. Lemma 3.3), by (3.18) we necessarily have:

$$P^+ \begin{pmatrix} V_0 \\ W_0 \end{pmatrix} \in \mathbb{C} \times \{0\}.$$

But the range of P^+ is $\mathbb{C}\binom{1}{(1+\mathrm{i})/\sqrt{2}}$ so that $P^+\binom{V_0}{W_0} = 0$. Finally,

$$\begin{pmatrix} V \\ V' \end{pmatrix}(Z) = \mathrm{e}^{-\frac{(1+\mathrm{i})}{\sqrt{2}}Z} \begin{pmatrix} V_0 \\ W_0 \end{pmatrix} + G * f. \tag{3.23}$$

With Eqs. (3.22) and (3.23), there exist C_1, σ_1 positive such that

$$\big\| \langle u \rangle \big\|_{H^1(Z>R)} \leqslant C_1\big(\exp(-\sigma_1 R) + \|\nabla u^*\|_{L^2(\widetilde{\omega}^{R/2})}\big).$$

This allows us to state:

Proposition 3.4. *There exist $C > 0$, $\sigma > 0$, such that for all $R > 0$,*

$$\|u\|_{H^1(\widetilde{\omega}^R)} \leqslant C\big(\exp(-\sigma R) + \|\nabla u^*\|_{L^2(\widetilde{\omega}^{R/2})}\big). \tag{3.24}$$

3.2.3. Control of high-order derivatives

Proposition 3.5. *For all $R > 0$, for all $m \geqslant 0$, $(u, p) \in H^m(\widetilde{\omega}^{R+1})^4$ with estimate*

$$\|u\|_{H^m(\widetilde{\omega}^{R+1})} + \|\nabla p\|_{H^m(\widetilde{\omega}^{R+1})} \leqslant C_m \|u\|_{H^1(\widetilde{\omega}^R)}.$$

Thanks to Proposition 3.4, it will then be enough to show the exponential decay of function $f(R) = \|\nabla u\|_{L^2(\widetilde{\omega}^R)}$. In order to prove this proposition, we use regularity properties of the Stokes operator. The following lemma is classical in the case of Dirichlet boundary conditions, and extends to our periodic case.

Lemma 3.6. *Let $u, \tau, f \in \mathcal{C}^\infty(\widetilde{\omega}^+)$ solution of*

$$\begin{aligned} \Delta u = \nabla \tau + f \quad & \text{in } \widetilde{\omega}^+, \\ \nabla \cdot u = 0 \quad & \text{in } \widetilde{\omega}^+, \\ u \text{ 1-periodic} \quad & \text{in } (X, Y). \end{aligned} \tag{3.25}$$

Then, for all $s \geqslant 1$, $\delta \in (0, s)$, for all $m \geqslant 0$ and $q \geqslant 1$,

$$\|u\|_{m+2,q,s,s+1} + \|\nabla \tau\|_{m,q,s,s+1} \leqslant C\big(\|f\|_{m,q,s-\delta,s+1+\delta} + \|u\|_{1,q,s-\delta,s+1+\delta}\big),$$

C being independent of s.

Proof. We show the estimate for $m = 0$, the general case is proved with an induction involving the same ideas. It is enough to prove the result for $s = 1$. We can always come back to this case by the change of variable $Z \to Z - \xi$. This automatically implies that the constant C of the lemma is independent of s.

Let $\delta > 0$ fixed. Let $0 \leqslant \psi \leqslant 1$ be a $\mathcal{C}^\infty$ function on $\mathbb{R}^3$ such that

$$\psi = 1 \quad \text{in } K = \left[-\frac{1}{2}, \frac{3}{2}\right]^2 \times \left[1 - \frac{\delta}{2}, 2 + \frac{\delta}{2}\right],$$
$$\psi = 0 \quad \text{outside a } \mathcal{C}^\infty \text{ open set } O \text{ with } K \subset O \Subset]-1, 2[^2 \times]1 - \delta, 2 + \delta[.$$

Following [10, p. 309], we then set $w = \psi u$, $q = \psi \tau$. (w, q) is then solution of:

$$\begin{aligned} &\Delta w = \nabla q + \tilde{f} + \widetilde{F} \quad \text{in } O, \\ &\nabla \cdot w = \tilde{g} \quad \text{in } O, \\ &w = 0 \quad \text{at } \partial O, \end{aligned} \tag{3.26}$$

where

$$\tilde{f} = \psi f, \qquad \widetilde{F} = 2\nabla\psi \cdot \nabla u + u\Delta\psi - \tau\nabla\psi, \qquad \tilde{g} = \nabla\psi \cdot u.$$

Using classical regularity results on Stokes equation in a $\mathcal{C}^\infty$ bounded domain, we get:

$$\|w\|_{W^{2,q}(0)} + \|\nabla q\|_{L^q(0)} \leqslant C\big(\|\tilde{f}\|_{L^q(0)} + \|\widetilde{F}\|_{L^q(0)} + \|\tilde{g}\|_{W^{1,q}(0)} + \|w\|_{L^q(0)}\big).$$

But it is easy to see that

$$\begin{aligned} &\|u\|_{2,q,1,2} \leqslant \|w\|_{W^{2,q}(0)}, \\ &\|\nabla\tau\|_{0,q,1,2} \leqslant \|u\|_{L^q(]-1,2[^2\times]1-\delta,2+\delta[)} \leqslant C\|u\|_{0,q,1-\delta,2+\delta}, \end{aligned}$$

as u is periodic in X, Y. From expressions of $\tilde{f}, \widetilde{F}, \tilde{g}$, we also get the inequality:

$$\begin{aligned} &\|\tilde{f}\|_{L^q(0)} + \|\widetilde{F}\|_{L^q(0)} + \|\tilde{g}\|_{W^{1,q}(0)} \\ &\quad \leqslant C\big(\|f\|_{0,q,1-\delta,2+\delta} + \|u\|_{1,q,1-\delta,2+\delta} + \|\tau\|_{0,q,1-\delta,2+\delta}\big). \end{aligned}$$

Up to modify τ by adding a suitable constant, we can suppose (see [10, p. 180])

$$\|\tau\|_{0,q,1-\delta,2+\delta} \leqslant C\big(\|f\|_{0,q,1-\delta,2+\delta} + \|u\|_{1,q,1-\delta,2+\delta}\big).$$

It leads to the desired inequality. □

Proof of Proposition 3.5. We use Lemma 3.6 and classical regularity arguments for stationary Navier–Stokes equations. We only treat the case $m = 0$, the general case is based on an induction involving the same ideas. Equations satisfied by u can be written:

$$-\Delta u + \nabla p = f \quad \text{in } \widetilde{\omega}^+,$$
$$\nabla \cdot u = 0 \quad \text{in } \widetilde{\omega}^+, \tag{3.27}$$

where $f = -(\mathbf{e} \times u + (u \cdot \nabla u))$. We are going to estimate u, p on $\widetilde{\omega}^{R,R+1}$ for different R. As already mentioned, thanks to the invariance by translation along Z, all constants appearing in the inequalities will be independent of R. Let $R > 0$. First,

$$\|\mathbf{e} \times u\|_{0,3/2,R,R+1} \leqslant \|u\|_{0,3/2,R,R+1} \leqslant \|u\|_{0,2,R,R+1} \leqslant \|u\|_{1,2,R,R+1}.$$

Then,

$$\begin{aligned}\big\|(u \cdot \nabla u)\big\|_{0,3/2,R,R+1} &\leqslant C_1 \|u\|_{0,6,R,R+1} \|\nabla u\|_{0,2,R,R+1} \leqslant C_2 \|u\|_{1,2,R,R+1} \|\nabla u\|_{L^2(\widetilde{\omega})} \\ &\leqslant C_2 \|u\|_{1,2,R,R+1},\end{aligned}$$

where C_2 depends on $\|u\|_{H^1(\widetilde{\omega})}$, which is finite by Proposition 3.4. Finally, $f \in L^{3/2}(\widetilde{\omega}^{R,R+1})^3$, and by Lemma 3.6, for $\delta > 0$, for all $R > 0$,

$$\|u\|_{2,3/2,R+\delta,R+1-\delta} \leqslant C\big(\|f\|_{0,3/2,R,R+1} + \|u^*\|_{1,3/2,R,R+1}\big) \leqslant C\|u\|_{1,3/2,R,R+1}.$$

By Sobolev injection, we deduce from that

$$\|u\|_{0,\infty,R+\delta,R+1-\delta} \leqslant C\|u\|_{1,2,R,R+1}.$$

We may now iterate the process and get an improved regularity on f. If we note $R_1 = R + \delta$, $R_2 = R + 1 - \delta$, we have of course,

$$\|\mathbf{e} \times u\|_{0,2,R_1,R_2} \leqslant C\|u\|_{1,2,R,R+1},$$
$$\big\|(u \cdot \nabla u)\big\|_{0,2,R_1,R_2} \leqslant C\|u\|_{0,\infty,R,R+1}\|u\|_{0,2,R_1,R_2} \leqslant C\|u\|_{1,2,R,R+1},$$

and applying again Lemma 3.6, we obtain:

$$\|u\|_{2,2,R_1+\delta,R_2-\delta} + \|\nabla p\|_{0,2,R_1+\delta,R_2-\delta} \leqslant C\|u\|_{1,2,R,R+1}.$$

With $\delta = 1/8$, we get:

$$\|u\|_{2,2,R+1/4,R-1/4} + \|\nabla p\|_{0,2,R+1/4,R-1/4} \leqslant C\|u\|_{1,2,R,R+1}.$$

Using Eq. (3.2) with $R := R + k/2$, $k = 1, 2, \ldots$, and summing over k, we get the result. $\square$

3.2.4. Exponential decay

It remains to show the exponential decay of $\|\nabla u^*\|_{L^2(\widetilde{\omega}^R)}$ (as R goes to infinity).

Proposition 3.7. *There exists $R_1, C, \sigma > 0$ such that*

$$\forall R \geqslant R_1, \quad \|\nabla v^*\|_{L^2(\widetilde{\omega}^R)} \leqslant C \exp(-\sigma R).$$

Proof. This proposition is close to Lemma 4.4 of [11] relative to a flow in a semi-infinite straight channel. We detail its proof for the sake of completeness. An energy estimate on (3.14), (3.15) gives, for all $R > 0$,

$$\begin{aligned}\int_{\widetilde{\omega}^R} |\nabla u^*|^2 = &-\int_{\Sigma(R)} \mathrm{d}X\,\mathrm{d}Y\left(-p^* u_3^* + \frac{1}{2}\frac{\partial |u^*|^2}{\partial Z} - \frac{1}{2}|u^*|^2\big(u_3^* + \langle u_3\rangle\big)\right)\\ &-\int_{\widetilde{\omega}^R} \big(u^* \cdot \nabla\langle u\rangle\big)\cdot u^*\,\mathrm{d}\mathbf{X},\end{aligned} \tag{3.28}$$

where we denote $\Sigma(R)$ the cross-section at $Z = R$. Let $f(R) = \int_{\widetilde{\omega}^R} |\nabla u^*|^2$.

$$\begin{aligned}\left|\int_{\widetilde{\omega}^R} \big(u^*.\nabla\langle u\rangle\big)\cdot u^*\right| &= \left|\int_R^{+\infty} \mathrm{d}Z\left(\int_{\Sigma(Z)} \mathrm{d}X\,\mathrm{d}Y\big(u^*\cdot\nabla\langle u\rangle\big)\cdot u^*\right)\right|\\ &\leqslant \left(\sup_{t\geqslant R}\left(\int_{\Sigma(t)} \big|\nabla\langle u\rangle\big|^2\right)^{1/2}\right)\left(\int_R^{+\infty} \mathrm{d}Z\left(\int_{\Sigma(Z)} \mathrm{d}X\,\mathrm{d}Y|u^*|^4\right)^{1/2}\right)\\ &\leqslant C\left(\sup_{t\geqslant R}\left(\int_{\Sigma(t)} \big|\nabla\langle u\rangle\big|^2\right)^{1/2}\right) f(R)\end{aligned}$$

(we have used the Sobolev injection $H^1 \hookrightarrow L^4$ in space dimension 2). Now, thanks to Proposition 3.5 there exists R_1 such that: $\forall R \geqslant R_1$,

$$\sup_{t\geqslant R}\left(\int_{\Sigma(t)} \mathrm{d}X\,\mathrm{d}Y\big|\nabla\langle u\rangle\big|^2\right)^{1/2} \leqslant \frac{1}{2C}.$$

Back to (3.28), we obtain:

$$\begin{aligned}\frac{1}{2} f(R) &\leqslant \int_{\Sigma(R)} \big(-p^* u_3^*\big) + \int_{\Sigma(R)} \left(\frac{1}{2}\frac{\partial |u^*|^2}{\partial Z}\right) + \int_{\Sigma(R)} \left(-\frac{1}{2}|u^*|^2\big(u_3^* + \langle u_3\rangle\big)\right)\\ &= I_1(R) + I_2(R) + I_3(R).\end{aligned} \tag{3.29}$$

- Using Proposition 3.5, we obtain easily, for R large enough,

$$\left|I_3(R)\right| \leqslant \int\limits_{\Sigma(R)} |u^*|^2 \leqslant \int\limits_{\Sigma(R)} |\nabla u^*|^2 = -f'(R).$$

- $$\left|\int\limits_t^{+\infty} I_2(R)\,\mathrm{d}R\right| = \int\limits_{\Sigma(t)} \frac{|u^*|^2}{2} \leqslant \int\limits_{\Sigma(t)} \frac{|\nabla u^*|^2}{2} = -\frac{1}{2} f'(t).$$

- Integral $I_3(R)$ is a bit more difficult to handle, as it is not directly quadratic in u^*.

Let $t > 0$. We introduce a solution ξ to the problem:

$$\begin{cases} \nabla \cdot \xi = u_3^*, \\ \xi \in H_0^1(\omega_{t,t+1})^3, \\ \|\xi\|_{H_0^1(\omega_{t,t+1})^3} \leqslant C_0 \|u_3^*\|_{L^2(\omega_{t,t+1})}. \end{cases} \tag{3.30}$$

As $\int_{\widetilde{\omega}_{t,t+1}} u_3^* = 0$, it is classical result that such a solution exist, with a constant C_0 independant of t (see [10]). Then we get, through an integration by parts,

$$\int\limits_t^{t+1} I_1(R)\,\mathrm{d}R = \int\limits_{\widetilde{\omega}_{t,t+1}} (\nabla p^* \cdot \xi).$$

Now, we multiply (3.14) by ξ and integrate. This leads to:

$$\int\limits_{\widetilde{\omega}_{t,t+1}} (\nabla p^* \cdot \xi) = -\int\limits_{\widetilde{\omega}_{t,t+1}} (\nabla u^* \cdot \nabla \xi) - \int\limits_{\widetilde{\omega}_{t,t+1}} (\mathbf{e} \times u^*)\xi - \int\limits_{\widetilde{\omega}_{t,t+1}} (u^* \cdot \nabla u^*)^* \xi. \tag{3.31}$$

We get, for t large enough,using again Proposition 3.5 and the estimate on ξ.

$$\int\limits_t^{t+1} I_1(R)\,\mathrm{d}R \leqslant C \int\limits_{\widetilde{\omega}_{t,t+1}} |\nabla u^*|^2,$$

with C independent of t. Thus, we finally obtain:

$$\int\limits_t^{+\infty} I_1(R)\,\mathrm{d}R \leqslant C f(t).$$

By integration of (3.29) from t to infinity, t large enough, and thanks to above bounds, it yields an integro-differential inequality of type

$$\int_t^{+\infty} f(R)\,\mathrm{d}R + C_1 f'(t) \leqslant C_2 f(t),$$

where the C_i's are positive constants. We conclude with a Gronwall's type lemma (cf. [11]). □

Gathering Propositions 3.5 and 3.7, we state

Corollary 3.8. $\exists R_1 > 0,\ C, \sigma > 0$, *such that*: $\forall Z > R_1,\ \forall \alpha \in \mathbb{Z}^3,\ \forall \mathbf{X} = (X, Y, Z)$,

$$\left|D^\alpha u(\mathbf{X})\right| + \left|D^\alpha \nabla p(\mathbf{X})\right| \leqslant C \exp(-\sigma Z).$$

3.3. Uniqueness of the solution

We want to show that there is at most one solution $\tilde{u}$, $\tilde{p}$ of (BL). Let $(\tilde{u}^1, \tilde{p}^1)$, $(\tilde{u}^2, \tilde{p}^2)$ be two solutions. We have $\tilde{u}^i = U_E + v^i$, $i = 1, 2$, with, for all φ in $\mathcal{V}$,

$$\int_{\widetilde{\omega}} \left(\mathbf{e} \times v^i\right) \cdot \varphi + \int_{\widetilde{\omega}} \left(U_E \cdot \nabla v^i\right) \cdot \varphi + \int_{\widetilde{\omega}} \left(v^i \cdot \nabla U_E\right) \cdot \varphi + \int_{\widetilde{\omega}} \left(v^i \cdot \nabla v^i\right) \cdot \varphi + \int_{\widetilde{\omega}} \nabla v^i \cdot \nabla \varphi$$
$$= \begin{pmatrix} -\mathbf{u}^\perp \\ 0 \end{pmatrix} \cdot \int_{\widetilde{\omega}^-} \varphi + \left[\frac{\partial U_E}{\partial Z}\right]\Big|_\Sigma \cdot \int_\Sigma \varphi. \tag{3.32}$$

To prove uniqueness, we would like to replace φ in $\mathcal{V}$ by $w = v^2 - v^1$ in (3.32). We will use the fact that v^i has a good behaviour at infinity to enlarge the space of test functions.

As w in V, there exists $(\varphi_n) \in \mathcal{V}$ which converges to w in V. Of course, (3.32) is satisfied with $\varphi = \varphi_n$ for all n, and we wish to pass to the limit. It is clear that

$$\int \nabla v^i \cdot \nabla \varphi_n \xrightarrow[n \to +\infty]{} \int \nabla v^i \cdot \nabla w,$$

$$\left|\begin{pmatrix} -\mathbf{u}^\perp \\ 0 \end{pmatrix} \cdot \int_{\widetilde{\omega}^-} (\varphi_n - w) + \left[\frac{\partial U_E}{\partial Z}\right]\Big|_\Sigma \cdot \int_\Sigma (\varphi_n - w)\right| \leqslant C \left\|\nabla(\varphi_n - w)\right\|_{L^2(\widetilde{\omega}^-)},$$

by the "trace" Theorem and Poincaré's inequality. It goes to zero as n goes to infinity.

It remains to treat the other volume integrals. Each of them can be divided into

$$\int_{\widetilde{\omega}} = \int_{\widetilde{\omega} \cap \{Z < R\}} + \int_{\widetilde{\omega}^R}, \quad R > 0.$$

- The convergence of the first integral comes from the fact that $\varphi_n \to \varphi$ strongly in $L^2(\widetilde{\omega} \cap \{Z > R\})$ thanks to Rellich's Theorem.

- As for the second integral, we only treat the convergence of $\int_{\widetilde{\omega}^R}(\mathbf{e}\times v^i)\cdot\varphi_n$ (the others can be handled in the same way),

$$\begin{aligned}
&\left|\int\limits_{\widetilde{\omega}^R}\big(\mathbf{e}\times v^i\big)(\varphi_n-w)\right| \\
&\quad=\int\limits_{(0,1)^2}\mathrm{d}X\,\mathrm{d}Y\int\limits_R^{+\infty}\mathrm{d}Z\big|\mathbf{e}\times v^i\big|\left|\int\limits_R^{Z}\frac{\partial(\varphi_n-w)}{\partial\theta}\,\mathrm{d}\theta+(\varphi_n-w)(X,Y,R)\right| \\
&\quad\leqslant\left(\sup_{X,Y\in(0,1)^2}\int\limits_R^{+\infty}\mathrm{d}Z\big|v^i\big|(X,Y,Z)\sqrt{Z-R}\right)\int\limits_{(0,1)^2}\mathrm{d}X\,\mathrm{d}Y\left(\int\limits_R^{+\infty}\mathrm{d}Z\big|\nabla(\varphi_n-w)\big|^2\right)^{1/2} \\
&\qquad+\left(\sup_{X,Y\in(0,1)^2}\int\limits_R^{+\infty}\mathrm{d}Z\big|v^i\big|(X,Y,Z)\right)\int\limits_{(0,1)^2}\mathrm{d}X\,\mathrm{d}Y\big|(\varphi_n-w)(X,Y,R)\big|.
\end{aligned}$$

Thus

$$\left|\int\limits_{\widetilde{\omega}^R}\big(\mathbf{e}\times v^i\big)(\varphi_n-w)\right|\leqslant C_1\big\|\nabla(\varphi_n-w)\big\|_{L^2(\widetilde{\omega}^R)}+C_2\big\|\nabla(\varphi_n-w)\big\|_{L^2(\widetilde{\omega}\cap\{Z<R\})}.$$

Thus, all terms pass to the limit and we may replace φ by w in (3.32). To end the proof is then easy (cf. [19] for instance), and leads to uniqueness under a smallness assumption on $[\partial U_E/\partial Z]$, i.e., on $|\mathbf{u}|$.

Let us gather all the results of this section:

(1) We have proved existence of a variational solution of (BL) through a Galerkin scheme.
(2) We have proved exponential decay of such a solution.
(3) Using this decay, we have proved uniqueness of such a solution.

This ends the proof of Theorem 1.1.

4. The interior term

The aim of this section is to prove Theorem 1.3, Section 1. It involves the function P, for which a formal expression is given in Section 2: $P=\widetilde{P}+\overline{P}$, where $\widetilde{P}$ is linked to the lower boundary layer, and $\overline{P}$ to the top one. As it has been the case up to now, we will focus on the bottom layer and operator $\widetilde{P}$, as similar definition and bounds hold for $\overline{P}$.

Let us take U_∞ as given in Theorem 1.1. We define:

$$\forall \mathbf{u} \in \mathbb{R}^2,\ |\mathbf{u}| \leqslant U_\infty, \quad P(\mathbf{u}) = \int_{\widetilde{\omega}} \begin{pmatrix} \tilde{u}_2 \\ -\tilde{u}_1 \end{pmatrix} \mathrm{d}\mathbf{X},$$

where $\tilde{u}$ is solution of (BL). We first show that P is dissipative.

4.1. Proof of Proposition 1.2

We show that $\widetilde{P}$ is dissipative (the same proof applies to $\overline{P}$, so that $P = \widetilde{P} + \overline{P}$ is dissipative). Let $\mathbf{u} \leqslant U_\infty$, and $\tilde{u}$ the solution of (BL). Let $\tilde{v}$ defined on $\widetilde{\omega}$ by:

$$\tilde{v} = \tilde{u} \quad \text{on } \widetilde{\omega}^+, \qquad \tilde{v} = \tilde{u} - \begin{pmatrix} \mathbf{u} \\ 0 \end{pmatrix} \quad \text{on } \widetilde{\omega}^-.$$

The function $\tilde{v}$ is solution of:

$$\begin{aligned}
&\mathbf{e} \times \tilde{v} + \nabla \tilde{p} + \tilde{v} \cdot \nabla \tilde{v} - \begin{pmatrix} \mathbf{u} \\ 0 \end{pmatrix} \cdot \nabla \tilde{v} - \Delta \tilde{v} = 0 \quad \text{in } \widetilde{\omega}^-, \\
&\mathbf{e} \times \tilde{v} + \nabla \tilde{p} + \tilde{v} \cdot \nabla \tilde{v} - \Delta \tilde{v} = 0 \quad \text{in } \widetilde{\omega}^+, \\
&\nabla \cdot \tilde{v} = 0 \quad \text{in } \widetilde{\omega}^+ \cup \widetilde{\omega}^-, \\
&[\tilde{v}]|_\Sigma = \left[\frac{\partial \tilde{v}}{\partial Z} - \tilde{p}\mathbf{e} \right]\Big|_\Sigma = 0 \quad \text{on } \Sigma, \\
&\tilde{v} = -\begin{pmatrix} \mathbf{u} \\ 0 \end{pmatrix} \quad \text{on } \tilde{\gamma},\ \tilde{v} \text{ 1-periodic in } (X, Y).
\end{aligned} \tag{BL4}$$

A straightforward energy estimate on (BL4) leads to:

$$\int_{\tilde{\gamma}} \left(\tilde{p}\nu - \frac{\partial \tilde{v}}{\partial \nu} \cdot \tilde{v} \right) + \int_{\widetilde{\omega}} |\nabla \tilde{v}|^2 + \int_{\tilde{\gamma}} \tilde{v} \cdot \nu \sum \frac{\tilde{v}_i^2}{2} + \int_{\tilde{\gamma}} \left(\begin{pmatrix} \mathbf{u} \\ 0 \end{pmatrix} \cdot \nu \sum \frac{\tilde{v}_i^2}{2} \right) = 0 \tag{4.1}$$

which, replacing $\tilde{v}$ by its value on $\tilde{\gamma}$ (and using that $\int_{\tilde{\gamma}} \binom{\mathbf{u}}{0} \cdot \nu = \int_{\widetilde{\omega}} \nabla \cdot \tilde{v} = 0$), leads to:

$$-\int_{\tilde{\gamma}} \left(\tilde{p}\, \nu - \frac{\partial \tilde{v}}{\partial \nu} \right) \cdot \begin{pmatrix} \mathbf{u} \\ 0 \end{pmatrix} + \int_{\widetilde{\omega}} |\nabla \tilde{v}|^2 = 0.$$

On the other hand,

$$\widetilde{P}(\mathbf{u}) \cdot \mathbf{u} = \left(\int_{\widetilde{\omega}} \mathbf{e} \times \tilde{v} \right) \cdot \begin{pmatrix} \mathbf{u} \\ 0 \end{pmatrix} = \int_{\widetilde{\omega}} (\nabla \tilde{p} - \Delta \tilde{v} + \tilde{v} \nabla \tilde{v}) \cdot \begin{pmatrix} \mathbf{u} \\ 0 \end{pmatrix} + \int_{\widetilde{\omega}^-} \left(\begin{pmatrix} \mathbf{u} \\ 0 \end{pmatrix} \cdot \nabla \tilde{v} \right) \cdot \begin{pmatrix} \mathbf{u} \\ 0 \end{pmatrix}.$$

Through an integration by parts, we obtain:

$$\widetilde{P}(\mathbf{u})\cdot\mathbf{u} = \int_{\tilde{\gamma}} \left(\tilde{p}\nu + \frac{\partial \tilde{v}}{\partial \nu}\right)\cdot\begin{pmatrix}\mathbf{u}\\ 0\end{pmatrix} = \int_{\widetilde{\omega}} |\nabla v|^2 \geqslant 0.$$

This ends the proof of the proposition.

We want to show existence of regular solutions in short time to system (Int). Therefore, we need to get bounds on $\widetilde{P}(u)$ and $\widetilde{P}(u) - \widetilde{P}(u')$.

4.2. *Bound on* $\widetilde{P}(u)$

The goal of this section is to prove:

Proposition 4.1. *For all $m \geqslant 3$, there exists an increasing function $\varphi_m \in C(\mathbb{R}^+, \mathbb{R}^+)$, and $\delta_m > 0$, such that, for all $T > 0$ and $u \in L^\infty(0,T; H^m(\mathbb{T}^2))^2$,*

$$\sup |u| \leqslant \delta_m \;\Rightarrow\; \big\|\widetilde{P}(u)\big\|_{L^\infty(0,T;H^m)^2} \leqslant \varphi_m\big(\|u\|_{L^\infty(0,T;H^m)^2}\big).$$

Proof. The proof of this proposition is divided into two steps:

(1) We first bound the norm of $\widetilde{P}(u)$ in spaces $L^\infty(0,T;H^m(\mathbb{T}^2))^2$ by the norm of $\nabla\tilde{u}$ in spaces $L^\infty(0,T;H^m(\mathbb{T}^2;L^2(\widetilde{\omega})))^9$.
(2) We then show the appropriate control on $\nabla\tilde{u}$, through energy estimates performed on the auxiliary system (BL2).

4.2.1. *Notations*

In what follows, we will skip variable t, as it is only a parameter in the inequalities to be proved.

We will consider two types of functions: $w = w(x,y)$ defined on $\mathbb{T}^2$ and $\widetilde{w} = \widetilde{w}(x,y,X,Y,Z)$, defined on $\mathbb{T}^2 \times \widetilde{\omega}$. To lighten notations, we set, for any domains Ω, Ω',

$$\|\cdot\|_{m,\Omega} := \text{the norm in } H^m(\Omega),$$
$$\|\cdot\|_{m,\Omega,s,\Omega'} := \text{the norm in } H^m\big(\Omega; H^s(\Omega')\big).$$

Finally, we will always use the notation ∇ for $\nabla_{\mathbf{X}}$, ∂^α for ∂_x^α, $\alpha > 0$.

Let $u \in H^m(\mathbb{T}^2)^2$, with $\sup|u| \leqslant U_\infty$. We associate to $u(x,y)$ the solution $\tilde{u}(x,y,\cdot)$ of (BL) with $\mathbf{u} = u(x,y)$. Variables x and y playing symmetric roles, it is then enough to control the quantities

$$\int_{\widetilde{\omega}} \mathbf{e}\times\partial^\alpha\tilde{u}, \quad \alpha = 0,\dots,m.$$

The first idea is to reduce the study of $\widetilde{P}(u)$ to the study of $\nabla\tilde{u}$, on which we can get energy estimates:

Lemma 4.2. *For all $m \geqslant 3$, we have*:

$$\|\widetilde{P}(u)\|^2_{m,\mathbb{T}^2} \leqslant C_{1,m}\|\nabla\tilde{u}\|^2_{m,\mathbb{T}^2,0,\widetilde{\omega}} + C_{2,m}\|\nabla\tilde{u}\|^4_{m,\mathbb{T}^2,0,\widetilde{\omega}}.$$

Proof. We write, for all $\alpha \geqslant 0$,

$$\int_{\widetilde{\omega}} \mathbf{e}\times\partial^\alpha\tilde{u}(x,y,\cdot) = \int_{\widetilde{\omega}^-} \mathbf{e}\times\partial^\alpha\tilde{u}(x,y,\cdot) + \int_{\widetilde{\omega}^+} \mathbf{e}\times\partial^\alpha\tilde{u}(x,y,\cdot).$$

First, using Poincaré's inequality,

$$\left|\int_{\widetilde{\omega}^-} \mathbf{e}\times\partial^\alpha\tilde{u}(x,y,\cdot)\right| \leqslant \|\partial^\alpha\tilde{u}\|_{0,\widetilde{\omega}^-} \leqslant C\|\partial^\alpha\nabla\tilde{u}\|_{0,\widetilde{\omega}^-} \leqslant C\|\partial^\alpha\nabla\tilde{u}\|_{0,\widetilde{\omega}}. \tag{4.2}$$

Then,

$$\left|\int_{\widetilde{\omega}^+} \mathbf{e}\times\partial^\alpha\tilde{u}(x,y,\cdot)\right| = \left|\int_0^{+\infty} \mathbf{e}\times\partial^\alpha\langle\tilde{u}\rangle(x,y,\cdot)\right| \leqslant \left\|\partial^\alpha\langle\tilde{u}\rangle(x,y,\cdot)\right\|_{L^1(\mathbb{R}^+)}. \tag{4.3}$$

It remains to evaluate the L^1 norm of $\langle\tilde{u}\rangle(x,y,\cdot)$. We use notations similar to those of Lemma 3.3 and Proposition 3.4:

$$V(x,y,Z) = \langle\tilde{u}_1\rangle(x,y,Z) + \mathrm{i}\langle\tilde{u}_2\rangle(x,y,Z),$$
$$F(x,y,Z) = \big\langle(\tilde{u}^*\cdot\nabla\tilde{u}^*)_1\big\rangle(x,y,Z) + \mathrm{i}\big\langle(\tilde{u}^*\cdot\nabla\tilde{u}^*)_2\big\rangle(x,y,Z),\ \ldots.$$

We differentiate formula (3.23) with respect to x, and obtain: for all $Z > 0$,

$$\begin{aligned}\begin{pmatrix}\partial^\alpha V(x,y,Z)\\ \partial^\alpha\partial_Z V(\cdot,Z)\end{pmatrix} &= \mathrm{e}^{\frac{-(1+\mathrm{i})}{\sqrt{2}}Z}\begin{pmatrix}\partial^\alpha V_0(x,y)\\ \partial^\alpha W_0(x,y)\end{pmatrix} + G*\partial^\alpha f(x,y,Z)\\ &= \mathrm{e}^{\frac{-(1+\mathrm{i})}{\sqrt{2}}Z}\begin{pmatrix}\partial^\alpha V(x,y,0)\\ \partial^\alpha W(x,y,0)\end{pmatrix} - \mathrm{e}^{\frac{-(1+\mathrm{i})}{\sqrt{2}}Z} G*\partial^\alpha f(x,y,0)\\ &\quad + G*\partial^\alpha f(x,y,Z).\end{aligned}$$

This implies, for all $1 \leqslant p \leqslant +\infty$,

$$\begin{aligned}\left\|\partial^\alpha\big\langle\tilde{u}(x,y,\cdot)\big\rangle\right\|_{L^p(\mathbb{R}^+)} \leqslant C\Big(&\left|\partial^\alpha\langle\tilde{u}\rangle(x,y,0)\right| + \left|G*\partial^\alpha f(x,y,0)\right|\\ &+ \left\|G*\partial^\alpha f(x,y,\cdot)\right\|_{L^p(\mathbb{R}^+)}\Big).\end{aligned}$$

We bound each term of the right-hand side, through

$$
\begin{aligned}
\left|\partial^\alpha\langle\tilde u\rangle(x,y,0)\right| &\leqslant C\left\|\tilde u(x,y,\cdot)\right\|_{L^2(\Sigma)} \leqslant C'\left\|\nabla\partial^\alpha\tilde u(x,y,\cdot)\right\|_{0,\widetilde\omega},\\
\left|G*\partial^\alpha f(x,y,0)\right| &\leqslant \|G\|_{L^\infty(\mathbb{R}^+)}\left\|\partial^\alpha f(x,y,\cdot)\right\|_{L^1(\mathbb{R}^+)},\\
\left\|G*\partial^\alpha f(x,y,\cdot)\right\|_{L^p(\mathbb{R}^+)} &\leqslant \|G\|_{L^p(\mathbb{R}^+)}\left\|\partial^\alpha f(x,y,\cdot)\right\|_{L^1(\mathbb{R}^+)}\\
&\leqslant \left\|\partial^\alpha(\tilde u^*.\nabla\tilde u^*)(x,y,\cdot)\right\|_{L^1(\widetilde\omega^+)}\\
&\leqslant C\sum_{\beta\leqslant\alpha}\left\|\nabla\partial^\beta\tilde u(x,y,\cdot)\right\|_{0,\widetilde\omega}\left\|\nabla\partial^{\alpha-\beta}\tilde u(x,y,\cdot)\right\|_{0,\widetilde\omega}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left\|\partial^\alpha\langle\tilde u(x,y,\cdot)\rangle\right\|_{L^p(\mathbb{R}^+)} \leqslant C\Bigg(&\left\|\nabla\partial^\alpha\tilde u(x,y,\cdot)\right\|_{0,\widetilde\omega}\\
&+\sum_{\beta\leqslant\alpha}\left\|\nabla\partial^\beta\tilde u(x,y,\cdot)\right\|_{0,\widetilde\omega}\left\|\nabla\partial^{\alpha-\beta}\tilde u(x,y,\cdot)\right\|_{0,\widetilde\omega}\Bigg)
\end{aligned}
\tag{4.4}
$$

with C depending on p. As $m\geqslant 3$, for all $w\in H^m(\mathbb{T}^2)$,

$$
\beta\leqslant\frac{m}{2}\ \Rightarrow\ \left\|\partial^\beta w\right\|_{L^\infty(\mathbb{T}^2)}\leqslant C_0\|w\|_{m,\mathbb{T}^2},
\tag{4.5}
$$

so that using (4.4) with $p=1$,

$$
\begin{aligned}
&\left\|\partial^\alpha\langle\tilde u(x,y,\cdot)\rangle\right\|_{L^1(\mathbb{R}^+)}\\
&\quad\leqslant C\left\|\nabla\partial^\alpha\tilde u(x,y,\cdot)\right\|_{0,\widetilde\omega}\\
&\qquad+2CC_0\|\nabla\tilde u\|_{m,\mathbb{T}^2,0,\widetilde\omega}\sum_{\alpha/2\leqslant\beta\leqslant\alpha}\left\|\nabla\partial^\beta\tilde u(x,y,\cdot)\right\|_{0,\widetilde\omega}\left\|\nabla\partial^{\alpha-\beta}\tilde u(x,y,\cdot)\right\|_{0,\widetilde\omega}.
\end{aligned}
\tag{4.6}
$$

With Eqs. (4.2), (4.3), (4.6), we get:

$$
\begin{aligned}
\left|\int_{\widetilde\omega}\mathbf{e}\times\partial^\alpha\tilde u(x,y,\cdot)\right| \leqslant{}& C_1\left\|\nabla\partial^\alpha\tilde u(x,y,\cdot)\right\|_{0,\widetilde\omega}\\
&+C_2\|\nabla\tilde u\|_{m,\mathbb{T}^2,0,\widetilde\omega}\sum_{\alpha/2\leqslant\beta\leqslant\alpha}\left\|\nabla\partial^\beta\tilde u(x,y,\cdot)\right\|_{0,\widetilde\omega}.
\end{aligned}
$$

Then:

$$
\begin{aligned}
\left|\int_{\widetilde\omega}\mathbf{e}\times\partial^\alpha\tilde u(x,y,\cdot)\right|^2 \leqslant{}& C_{1,m}\left\|\nabla\partial^\alpha\tilde u(x,y,\cdot)\right\|_{0,\widetilde\omega}^2\\
&+C_{2,m}\|\nabla\tilde u\|_{m,\mathbb{T}^2,0,\widetilde\omega}^2\sum_{\alpha/2\leqslant\beta\leqslant\alpha}\left\|\nabla\partial^\beta\tilde u(x,y,\cdot)\right\|_{0,\widetilde\omega}^2.
\end{aligned}
$$

Integrating in x, y for $\alpha=0,\dots,m$, and adding up, we get the result. □

With Lemma 4.2, the proof of Proposition 4.1 becomes a direct consequence of:

Lemma 4.3. *For all $m \geqslant 3$, there exists $\varphi_m \in C(\mathbb{R}^+, \mathbb{R}^+)$ an increasing function, and $\delta_m > 0$, such that for all $u \in H^m(\mathbb{T}^2)$,*

$$\sup |u| \leqslant \delta_m \;\Rightarrow\; \|\nabla \tilde{u}\|_{m,\mathbb{T}^2,0,\widetilde{\omega}} \leqslant \varphi_m\big(\|u\|_{m,\mathbb{T}^2}\big).$$

Proof. We may write:

$$\tilde{u}(x, y, \cdot) = U_E(x, y, \cdot) + v(x, y, \cdot),$$

where $U_E(x, y, \cdot)$ is given by (3.1) and $v(x, y, \cdot)$ is given by Theorem 3.2 (still for $\mathbf{u} = u(x, y)$). It is clear that we can work with v instead of $\tilde{u}$: we will carry energy estimates on $\nabla\partial^\alpha v(x, y, \cdot)$, thanks to (BL2), for $\alpha \leqslant m$.

Case $\alpha = 0$. Such an estimate has been carried in previous section for the approximate solution v^n. In the same way,

$$\big\|\nabla v(x, y, \cdot)\big\|^2_{0,\widetilde{\omega}} \leqslant C_1\big|u(x, y)\big|\big\|\nabla v(x, y, \cdot)\big\|_{0,\widetilde{\omega}} + C_2\big|u(x, y)\big|\big\|\nabla v(x, y, \cdot)\big\|^2_{0,\widetilde{\omega}}.$$

If $\sup |u| \leqslant \delta = C_2/2$, we get:

$$\big\|\nabla v(x, y, \cdot)\big\|_{0,\widetilde{\omega}} \leqslant 2C_1\big|u(t, x, y)\big|. \tag{4.7}$$

It implies $\|\nabla v\|^2_{0,\mathbb{T}^2,0,\widetilde{\omega}} \leqslant C\|u\|_{m,\mathbb{T}^2}$.

Case $\alpha \geqslant 1$. We differentiate α times Eqs. (BL2) with respect to x. We multiply by $\partial^\alpha v$ and integrate by parts:

$$\begin{aligned}
\big\|\nabla\partial^\alpha v(x, y, \cdot)\big\|^2_{0,\widetilde{\omega}} \leqslant{}& \bigg|\int_{\widetilde{\omega}} \partial^\alpha(v \cdot \nabla v) \cdot \partial^\alpha v(x, y, \cdot)\bigg| + \bigg|\int_{\widetilde{\omega}} \partial^\alpha(U_E \cdot \nabla v) \cdot \partial^\alpha v(x, y, \cdot)\bigg| \\
&+ \bigg|\int_{\widetilde{\omega}} \partial^\alpha(v \cdot \nabla U_E) \cdot \partial^\alpha v(x, y, \cdot)\bigg| + \big|\partial^\alpha u(x, y)\big| \int_{\widetilde{\omega}^-} \big|\partial^\alpha v(x, y, \cdot)\big| \\
&+ \big|\big[\partial^\alpha\partial_Z U_E(x, y, 0)\big]\big|_\Sigma \int_\Sigma \big|\partial^\alpha v(x, y, \cdot)\big| \\
={}& I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{4.8}$$

Let us bound each expression (we drop parameters x and y in these calculations to lighten notations).

As $\int (v \cdot \nabla\partial^\alpha v) \cdot \partial^\alpha v = 0$,

$$I_1 = \sum_{\beta=1}^{\alpha} \int_{\widetilde{\omega}} \big|\big(\partial^\beta v \cdot \nabla\partial^{\alpha-\beta} v\big) \cdot \partial^\alpha v\big| \leqslant \int_{\widetilde{\omega}^-} |\partial^\alpha v|^2 |\nabla v| + \sum_{\beta=1}^{\alpha-1} \int_{\widetilde{\omega}^-} \big|\big(\partial^\beta v \cdot \nabla\partial^{\alpha-\beta} v\big) \cdot \partial^\alpha v\big|$$

$$
\begin{aligned}
&+\int_{\widetilde{\omega}^+} |\partial^\alpha v|^2|\nabla v| + \sum_{\beta=1}^{\alpha-1}\int_{\widetilde{\omega}^+}\left|\left(\partial^\beta v\cdot\nabla\partial^{\alpha-\beta}v\right)\cdot\partial^\alpha v\right| \\
\leqslant{}& C\left(\|\nabla v\|_{0,\widetilde{\omega}}\|\partial^\alpha v\|_{1,\widetilde{\omega}}^2 + \sum_{\beta=1}^{\alpha-1}\left\|\partial^\beta v\right\|_{1,\widetilde{\omega}}\left\|\nabla\partial^{\alpha-\beta}v\right\|_{0,\widetilde{\omega}}\|\partial^\alpha v\|_{1,\widetilde{\omega}}\right) \\
&+\int_{\widetilde{\omega}^+} |\partial^\alpha v|^2|\nabla v| + \sum_{\beta=1}^{\alpha-1}\int_{\widetilde{\omega}^+}\left|\left(\partial^\beta v\cdot\nabla\partial^{\alpha-\beta}v\right)\cdot\partial^\alpha v\right|.
\end{aligned}
\tag{4.9}
$$

Now

$$
\begin{aligned}
\int_{\widetilde{\omega}^+} |\partial^\alpha v|^2|\nabla v| &\leqslant \sum_{k=0}^{+\infty}\int_{\widetilde{\omega}^{k,k+1}} |\partial^\alpha v|^2|\nabla v| \leqslant \sum_{k=0}^{+\infty}\|\nabla v\|_{0,\widetilde{\omega}^{k,k+1}}\|\partial^\alpha v\|^2_{L^4(\widetilde{\omega}^{k,k+1})} \\
&\leqslant C\|\nabla v\|_{L^2(\widetilde{\omega})}\sum_{k=0}^{+\infty}\|\partial^\alpha v\|^2_{1,\widetilde{\omega}^{k,k+1}} \leqslant C\|\nabla v\|_{0,\widetilde{\omega}}\|\partial^\alpha v\|^2_{1,\widetilde{\omega}}.
\end{aligned}
$$

Note that we have divided $\widetilde{\omega}$ into bounded "slices" $\widetilde{\omega}^{k,k+1}$, so as to use the Sobolev injection:

$$
\|w\|_{L^4(\widetilde{\omega}^{k,k+1})} \leqslant C\|w\|_{H^1(\widetilde{\omega}^{k,k+1})}, \quad C \text{ independent of } k.
$$

And similarly, we have:

$$
\begin{aligned}
\int_{\widetilde{\omega}^-}\left|\left(\partial^\beta v\cdot\nabla\partial^{\alpha-\beta}v\right)\cdot\partial^\alpha v\right| &\leqslant \sum_{k=0}^{+\infty}\|\partial^\alpha v\|_{L^4(\widetilde{\omega}^{k,k+1})}\left\|\partial^\beta v\right\|_{L^4(\widetilde{\omega}^{k,k+1})}\left\|\nabla\partial^{\alpha-\beta}v\right\|_{0,\widetilde{\omega}^{k,k+1}} \\
&\leqslant C\|\partial^\alpha v\|_{1,\widetilde{\omega}}\sum_{k=0}^{+\infty}\left(\left\|\partial^\beta v\right\|_{1,\widetilde{\omega}^{k,k+1}}\left\|\partial^{\alpha-\beta}v\right\|_{1,\widetilde{\omega}^{k,k+1}}\right) \\
&\leqslant C\|\partial^\alpha v\|_{1,\widetilde{\omega}}\left(\left\|\partial^\beta v\right\|_{1,\widetilde{\omega}}\left\|\partial^{\alpha-\beta}v\right\|_{1,\widetilde{\omega}}\right).
\end{aligned}
$$

It leads to

$$
I_1 \leqslant C\left(\|\nabla v\|_{0,\widetilde{\omega}}\|\partial^\alpha v\|^2_{1,\widetilde{\omega}} + \sum_{\beta=1}^{\alpha-1}\left\|\partial^\beta v\right\|_{1,\widetilde{\omega}}\left\|\partial^{\alpha-\beta}v\right\|_{1,\widetilde{\omega}}\|\partial^\alpha v\|_{1,\widetilde{\omega}}\right).
$$

With the case $\alpha = 0$, we get:

$$
I_1 \leqslant C_1\left(\sup|u|\|\partial^\alpha v\|^2_{1,\widetilde{\omega}} + \sum_{\beta=1}^{\alpha-1}\left\|\partial^\beta v\right\|_{1,\widetilde{\omega}}\left\|\partial^{\alpha-\beta}v\right\|_{1,\widetilde{\omega}}\|\partial^\alpha v\|_{1,\widetilde{\omega}}\right).
$$

Other integrals. With similar (and even simpler) manipulations we obtain:

$$I_2 \leqslant C_2\left(\|U_E\|_{L^\infty(\widetilde{\omega})}\|\partial^\alpha v\|_{1,\widetilde{\omega}}^2 + \sum_{\beta=1}^{\alpha}\left\|\partial^\beta U_E\right\|_{L^\infty(\widetilde{\omega})}\left\|\partial^{\alpha-\beta}\nabla v\right\|_{0,\widetilde{\omega}}\|\partial^\alpha v\|_{1,\widetilde{\omega}}\right),$$

$$I_3 \leqslant C_3\left(\|\nabla U_E\|_{L^\infty(\widetilde{\omega})}\|\partial^\alpha v\|_{1,\widetilde{\omega}}^2 + \sum_{\beta=1}^{\alpha}\left\|\nabla\partial^\beta U_E\right\|_{L^\infty(\widetilde{\omega})}\left\|\partial^{\alpha-\beta} v\right\|_{0,\widetilde{\omega}}\|\partial^\alpha v\|_{1,\widetilde{\omega}}\right),$$

$$I_4 + I_5 \leqslant C_4\left|\partial^\alpha u(x,y)\right|\|\nabla\partial^\alpha v\|_{0,\widetilde{\omega}}.$$

We report this in (4.8) to obtain: $\forall\alpha \leqslant m$,

$$\begin{aligned}\left\|\partial^\alpha\nabla v(x,y,\cdot)\right\|_{0,\widetilde{\omega}} \leqslant C\Bigg(&\sup|u|\left\|\partial^\alpha v(x,y,\cdot)\right\|_{1,\widetilde{\omega}}^2\\ &+\sum_{\beta=1}^{\alpha-1}\left\|\partial^\beta v(x,y,\cdot)\right\|_{1,\widetilde{\omega}}\left\|\partial^{\alpha-\beta} v(x,y,\cdot)\right\|_{1,\widetilde{\omega}}\left\|\partial^{\alpha} v(x,y,\cdot)\right\|_{1,\widetilde{\omega}}\\ &+\sum_{\beta=1}^{\alpha}\left|\partial^\beta u(x,y)\right|\left\|\partial^{\alpha-\beta} v(x,y,\cdot)\right\|_{1,\widetilde{\omega}}\\ &+\left|\partial^\alpha u(x,y)\right|\left\|\partial^{\alpha} v(x,y,\cdot)\right\|_{1,\widetilde{\omega}}\Bigg).\end{aligned} \tag{4.10}$$

Let us now prove by induction on α that: $\exists\delta > 0$, $\forall\alpha \leqslant m$, $\exists\,\varphi_\alpha \in C(\mathbb{R}^+,\mathbb{R}^+)$ an increasing function such that

$$\sup|u| \leqslant \delta \ \Rightarrow\ \sup_{k\leqslant\alpha}\left\|\partial^k v\right\|_{0,\mathbb{T}^2,1,\widetilde{\omega}} \leqslant \varphi_\alpha\left(\|u\|_{m,\mathbb{T}^2}\right).$$

The result is true for $\alpha = 0$, thanks to (4.4) with $p = 2$.

Let $\alpha \geqslant 1$, and assume that the result is true for $\alpha - 1$. From (4.4) with $p = 2$, we deduce easily that

$$\begin{aligned}\left\|\partial^\alpha v(x,y,\cdot)\right\|_{1,\widetilde{\omega}} \leqslant C\Bigg(&\left\|\nabla\partial^\alpha v(x,y,\cdot)\right\|_{0,\widetilde{\omega}}\\ &+\sum_{\beta=0}^{\alpha}\left\|\nabla\partial^\beta v(x,y,\cdot)\right\|_{0,\widetilde{\omega}}\left\|\nabla\partial^{\alpha-\beta} v(x,y,\cdot)\right\|_{0,\widetilde{\omega}}\Bigg),\end{aligned}$$

i.e.,

$$\|\partial^\alpha v(x,y,\cdot)\|_{1,\widetilde{\omega}} \leqslant C_1 \|\nabla\partial^\alpha v(x,y,\cdot)\|_{0,\widetilde{\omega}} + C_2 \sum_{\beta=1}^{\alpha-1} \|\nabla\partial^\beta v(x,y,\cdot)\|_{0,\widetilde{\omega}} \|\nabla\partial^{\alpha-\beta} v(x,y,\cdot)\|_{0,\widetilde{\omega}}$$

using that $\|\nabla v(x,y,\cdot)\|_{0,\widetilde{\omega}}$ is bounded by $C\sup|u|$. Now, the induction assumption leads to:

$$\|\partial^\alpha v(x,y,\cdot)\|_{1,\widetilde{\omega}} \leqslant C_1 \|\nabla\partial^\alpha v(x,y,\cdot)\|_{0,\widetilde{\omega}} + \phi_0\big(\|u\|_{m,\mathbb{T}^2}\big), \tag{4.11}$$

where $\phi_0 = C_2(\alpha-1)\varphi_{\alpha-1}^2$ is an increasing function. We may now "close" Eq. (4.10). Using $ab \leqslant \frac{1}{2\varepsilon^2}a^2 + \frac{\varepsilon^2}{2}b^2$, we get:

$$\begin{aligned}\|\nabla\partial^\alpha v(x,y,\cdot)\|_{0,\widetilde{\omega}}^2 \leqslant{}& C\left(\sup|u| + \frac{3}{2}\varepsilon^2\right)\|\partial^\alpha v(x,y,\cdot)\|_{1,\widetilde{\omega}}^2 \\ &+ \frac{C}{2\varepsilon^2}\left(\sum_{\beta=1}^{\alpha-1}\|\partial^\beta v(x,y,\cdot)\|_{1,\widetilde{\omega}}\|\partial^{\alpha-\beta} v(x,y,\cdot)\|_{1,\widetilde{\omega}}\right)^2 \\ &+ \frac{C}{2\varepsilon^2}\left(\sum_{\beta=1}^{\alpha}|\partial^\beta u(x,y)|\,\|\partial^{\alpha-\beta} v(x,y,\cdot)\|_{1,\widetilde{\omega}}\right)^2 \\ &+ \frac{C}{2\varepsilon^2}|\partial^\alpha u(x,y)|^2.\end{aligned} \tag{4.12}$$

Up to take δ and $\varepsilon > 0$ small enough, and using (4.11), we get:

$$\begin{aligned}\|\nabla\partial^\alpha v(x,y,\cdot)\|_{0,\widetilde{\omega}}^2 \leqslant{}& \phi_1\big(\|u\|_{m,\mathbb{T}^2}\big) + C_2\left(\sum_{\beta=1}^{\alpha-1}\|\partial^\beta v(x,y,\cdot)\|_{1,\widetilde{\omega}}\|\partial^{\alpha-\beta} v(x,y,\cdot)\|_{1,\widetilde{\omega}}\right)^2 \\ &+ C_3\left(\sum_{\beta=1}^{\alpha}|\partial^\beta u(x,y)|\,\|\partial^{\alpha-\beta} v(x,y,\cdot)\|_{1,\widetilde{\omega}}\right)^2 + C_4\big(|\partial^\alpha u(x,y)|\big)^2\end{aligned}$$

and using the induction assumption,

$$\|\nabla\partial^\alpha v(x,y,\cdot)\|_{0,\widetilde{\omega}}^2 \leqslant \phi_2\big(\|u\|_{m,\mathbb{T}^2}\big) + \phi_2\big(\|u\|_{m,\mathbb{T}^2}\big)\sum_{0\leqslant\beta\leqslant\alpha}|\partial^\beta u(x,y)|^2.$$

Integrating on $(x,y)\in\mathbb{T}^2$, we get the result. □

4.3. Bound on $\widetilde{P}(u) - \widetilde{P}(u')$

Computations on $\widetilde{P}(u) - \widetilde{P}(u')$ in $L^\infty(0,T;\ H^m)^2$ are of the same type as above (and even simpler to carry). Indeed, if v and v' are the solutions of Eqs. (BL2) associated to u and u', the difference $w = v - v'$ is solution of a system similar to (BL2) (and even simpler), namely:

$$\begin{aligned}
&\mathbf{e}\times w + \nabla\tilde{q} + U_E\cdot\nabla w + (U_E - U'_E)\cdot\nabla w + v\cdot\nabla\big(U_E - U'_E\big)\\
&\quad + w\cdot\nabla U'_E + v\cdot\nabla w + w\cdot\nabla w' - \Delta w = \begin{pmatrix} -(u-u')^\perp \\ 0\end{pmatrix} \quad \text{in } \widetilde{\omega}^-,\\
&\mathbf{e}\times w + \nabla\tilde{q} + U_E\cdot\nabla w + \big(U_E - U'_E\big)\cdot\nabla w + v\cdot\nabla\big(U_E - U'_E\big)\\
&\quad + w\cdot\nabla U'_E + v\cdot\nabla w + w\cdot\nabla w' - \Delta w = 0 \quad \text{in } \widetilde{\omega}^+,\\
&\nabla\cdot w = 0 \quad \text{in } \widetilde{\omega}^+\cup\widetilde{\omega}^-,\\
&[w]|_\Sigma = 0 \quad \text{at } \Sigma,\\
&\left[\frac{\partial w}{\partial Z} - \tilde{q}\mathbf{e}\right]\bigg|_\Sigma = -\left[\frac{\partial(U_E - U'_E)}{\partial Z}\right]\bigg|_\Sigma \quad \text{on } \Sigma,\\
&w = 0 \quad \text{on } \tilde{\gamma},\ w \text{ 1-periodic in } (X,Y).
\end{aligned}\tag{BL3}$$

Therefore, we state the following proposition without proof:

Proposition 4.4. *For all $m \geqslant 3$, there exists $\varphi_m \in C(\mathbb{R}^+\times\mathbb{R}^+, \mathbb{R}^+)$ and $\delta_m > 0$, such that, for all $T > 0$ and for all $u, u' \in L^\infty(0,T,H^m)^2$, $\sup|u| + \sup|u'| \leqslant \delta_m$ implies*

$$\big\|\widetilde{P}(u) - \widetilde{P}(u')\big\|_{L^\infty(H^m)} \leqslant \varphi_m\big(\|u\|_{L^\infty(H^m)}, \|u'\|_{L^\infty(H^m)}\big)\|u - u'\|_{L^\infty(H^m)},$$

where $L^\infty(H^m)$ stands for $L^\infty(0,T;H^m(\mathbb{T}^2))^2$.

4.4. Resolution of (Int)

It is now routine to construct a regular solution in short time to the system:

$$\begin{aligned}
&\partial_t\xi + u\cdot\nabla\xi + \operatorname{curl}\widetilde{P}(u) = 0 \quad \text{on } \mathbb{T}^2,\\
&\xi = \operatorname{curl} u, \quad \nabla\cdot u = 0 \quad \text{on } \mathbb{T}^2,\\
&u = u_0 \quad \text{at } t = 0,
\end{aligned}$$

under a smallness assumption on $\|u_0\|_\infty$. For instance, one may use the iterative scheme:

$$\partial_t\xi^{n+1} + u^n\cdot\nabla\xi^{n+1} + \operatorname{curl}\widetilde{P}(u^n) = 0, \tag{4.13}$$

$$\xi^{n+1} = \operatorname{curl} u^{n+1}, \quad \nabla\cdot u^{n+1} = 0, \tag{4.14}$$

$$u^{n+1}|_{t=0} = u^0. \tag{4.15}$$

Thanks to Propositions 4.1 and 4.4, if $\|u_0\|_\infty$ is small enough, one shows that

(1) there exists $T_1 > 0$, such that (u^n) is a bounded sequence in $L^\infty(0, T_1; H^m(\mathbb{T}^2))^2$,
(2) there exists $0 < T < T_1$, such that (u^n) is a Cauchy sequence in $L^\infty(0, T; H^m(\mathbb{T}^2))^2$.

For more about this type of schemes, see for instance [16] (with application to classical fluid mechanics systems).

As P and $\widetilde{P}$ share the same bounds, we have similarly existence of a regular solution to system (Int), under a smallness assumption on $\|u_0\|_\infty$. It ends the proof of Theorem 1.3.

5. Proof of convergence

It remains to prove the convergence result, i.e., Theorem 1.4. The general scheme of the proof is classical. On the basis of previous analysis, we build an approximate solution of the rotating fluid system. We then show that this approximation is close to an exact solution, through energy estimates. Such type of proof was already used in [13] for the flat case $\Omega = \mathbb{T}^2 \times (0, 1)$.

However, the presence of rough boundaries leads to some additional technicalities. For instance, the boundary layer terms $\tilde{u}^i(t, x, y, X, Y, Z)$ are not smooth for $(X, Y, Z) \in \widetilde{\omega}$, but on each side of the interface. This explains the introduction of integrals $I_R^{\varepsilon,-}$ and $I_R^{\varepsilon,+}$ at the end of Section 5.2. Moreover, as a general rule, we must be careful with the use of functional inequalities: indeed, such inequalities involve constants that depend on the domain, and thus may badly depend on ε.

5.1. The approximate solution

We will first construct an approximate solution of (0.3), (0.4), (1.1), based on the formal analysis of Section 2, and the results of Sections 3 and 4. Namely, we consider approximations:

$$\begin{aligned} u^\varepsilon_{\text{app}}(t, x, y, z) = {} & u^0(t, x, y, z) + \varepsilon u^1(t, x, y, z) + \tilde{u}^0\left(t, x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z}{\varepsilon}\right) \\ & + \varepsilon \tilde{u}^1\left(t, x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z}{\varepsilon}\right) + \bar{u}^0\left(t, x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z-1}{\varepsilon}\right) \\ & + \varepsilon \bar{u}^1\left(t, x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z-1}{\varepsilon}\right), \end{aligned} \tag{5.1}$$

$$\begin{aligned} p^\varepsilon_{\text{app}}(t, x, y, z) = {} & p^0(t, x, y, z) + \tilde{p}^0\left(t, x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z}{\varepsilon}\right) + \varepsilon \tilde{p}^1\left(t, x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z}{\varepsilon}\right) \\ & + \bar{p}^0\left(t, x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z-1}{\varepsilon}\right) + \varepsilon \bar{p}^1\left(t, x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z-1}{\varepsilon}\right), \end{aligned} \tag{5.2}$$

where profiles u^i, $\tilde{u}^i$, $\bar{u}^i$, p^i, $\tilde{p}^i$, $\bar{p}^i$ are defined below.

5.1.1. Definition of u^0, p^0, $\tilde{u}^0$, $\tilde{p}^0$, $\tilde{p}^1$

Let $m = 3$, T_m and δ_m given in Theorem 1.3, and $u_0 \in \dot{H}^m(\mathbb{T}^2)^2$ with $\|u_0\|_{L^\infty} \leqslant \delta_m$. Let u solution of (Int) with initial data u_0. We define:

$$\forall t \in (0,T),\ \forall (x,y,z) \in \Omega, \quad u^0(t,x,y,z) = \begin{pmatrix} u(t,x,y) \\ 0 \end{pmatrix}.$$

In agreement with Section 2, we set $u^0 = 0$ in $\Omega^\varepsilon - \Omega$. Let $\nabla p^0 = -\mathbf{e} \times u^0$. Let $\tilde{p}^0$ be defined by (2.13). Finally, let $(\tilde{u}^0(t,x,y,\cdot), \tilde{p}^1(t,x,y,\cdot))$ the unique solution of (BL) with $\mathbf{u} = u(t,x,y)$.

5.1.2. Definition of u^1, $\tilde{u}^1$

We define u^1 on Ω by:

$$\begin{aligned}
&-\mathbf{e} \times u^1 = \partial_t u^0 + u^0 \cdot \nabla u^0, \\
&\nabla \cdot u^1 = 0, \\
&u_3^1(\cdot, z=0) = -\int_{\widetilde{\omega}} \partial_x \tilde{u}_1^0 + \partial_y \tilde{u}_2^0.
\end{aligned}$$

$\tilde{u}^1$ is then chosen so that

$$\begin{aligned}
&\nabla_X \cdot \tilde{u}^1 = -\left(\partial_x \tilde{u}_1^0 + \partial_y \tilde{u}_2^0\right) \quad \text{in } \widetilde{\omega}, \\
&\left[\tilde{u}^1\right]\big|_\Sigma = -\left[u^1\right]\big|_\Sigma \quad \text{on } \Sigma.
\end{aligned}$$

Existence of $\tilde{u}^1$. Let $\tilde{\eta}$ 1-periodic in X, Y solution of:

$$\begin{aligned}
&\Delta_{\mathbf{X}} \tilde{\eta} = -\left(\partial_x \tilde{u}_1^0 + \partial_y \tilde{u}_2^0\right) \quad \text{in } \widetilde{\omega}, \\
&\left[\frac{\partial \tilde{\eta}}{\partial Z}\right]\bigg|_\Sigma = -\left[u_3^1\right]\big|_{z=0} \quad \text{on } \Sigma, \\
&\frac{\partial \tilde{\eta}}{\partial n} = 0 \quad \text{on } \tilde{\gamma}.
\end{aligned}$$

The compatibility condition,

$$\int_{\widetilde{\omega}} \left(\partial_x \tilde{u}_1^0 + \partial_y \tilde{u}_2^0\right) = \int_\Sigma \left[u_3^1\right]\big|_{z=0}$$

is fulfilled by definition of $\tilde{u}^1$, so that this system has a unique variational solution by Lax–Milgram Lemma. As $\partial_x \tilde{u}_1^0 + \partial_y \tilde{u}_2^0$ has exponential decay, $\nabla_X \tilde{\eta}$ will have exponential decay as Z goes to infinity. Let us then define $\tilde{v}$ by:

$$\tilde{v}_1 = -\big([u_1^1]_{z=0} - [\partial_X \tilde{\eta}]_\Sigma\big)\mathrm{e}^{-Z} \quad \text{in } \widetilde{\omega}^+, \tag{5.3}$$

$$\tilde{v}_2 = -\big([u_2^1]_{z=0} - [\partial_y \tilde{\eta}]_\Sigma\big)\mathrm{e}^{-Z} \quad \text{in } \widetilde{\omega}^+, \tag{5.4}$$

$$\tilde{v}_3 = 0 \quad \text{in } \widetilde{\omega}, \qquad \tilde{v} = 0 \quad \text{in } \widetilde{\omega}^-. \tag{5.5}$$

Then $\tilde{u}^1 = \tilde{v} + \nabla_{\mathbf{X}}\tilde{\eta}$ works.

Remark. Upper correctors $\bar{u}^i$, $\bar{p}^i$ are defined in the same way. In particular, we build $\bar{u}^1 = \bar{v} + \nabla\bar{\eta}$, with $\bar{\eta}$ solution of:

$$\Delta_{\mathbf{X}}\bar{\eta} = -\big(\partial_x \bar{u}_1^0 + \partial_y \bar{u}_2^0\big) \quad \text{in } \overline{\omega},$$
$$\left[\frac{\partial\bar{\eta}}{\partial Z}\right]\bigg|_\Sigma = -\big[u_3^1\big]\big|_{z=1} \quad \text{on } \Sigma,$$
$$\frac{\partial\bar{\eta}}{\partial n} = 0 \quad \text{on } \bar{\gamma}.$$

Note that in this case, the compatibility condition $\int_{\overline{\omega}}(\partial_x \bar{u}_1^0 + \partial_y \bar{u}_2^0) = \int_\Sigma [u_3^1]|_{z=1}$ is exactly Eq. (Int).

5.2. *Energy estimates*

Let u^ε a solution of (0.3), (0.4), (1.1), and $u^\varepsilon_{\mathrm{app}}$, $p^\varepsilon_{\mathrm{app}}$ given by (5.1), (5.2). $v^\varepsilon = u^\varepsilon - u^\varepsilon_{\mathrm{app}}$ and $q^\varepsilon = p^\varepsilon - p^\varepsilon_{\mathrm{app}}$ satisfy, for $i = 1, 2$,

$$\begin{aligned}
&\partial_t v^\varepsilon + \frac{\mathbf{e}\times v^\varepsilon}{\varepsilon} + \frac{\nabla q^\varepsilon}{\varepsilon} - \varepsilon\Delta v^\varepsilon + v^\varepsilon\cdot\nabla u^\varepsilon_{\mathrm{app}} + u^\varepsilon\cdot\nabla v^\varepsilon = R^\varepsilon \quad \text{in } \Omega\cup\Omega_i^\varepsilon,\\
&\nabla\cdot v^\varepsilon = r^\varepsilon \quad \text{in } \Omega\cup\Omega_i^\varepsilon,\\
&[v^\varepsilon]|_{\Sigma_i} = 0 \quad \text{at } \Sigma_i,\\
&\left[\varepsilon\frac{\partial v^\varepsilon}{\partial n} - \varepsilon^{-1}q^\varepsilon\vec{n}\right]\bigg|_{\Sigma_i} = \sigma^\varepsilon \quad \text{at } \Sigma_i,\\
&v^\varepsilon = \varphi^\varepsilon \quad \text{at } \Gamma_1^\varepsilon\cup\Gamma_2^\varepsilon,
\end{aligned} \tag{5.6}$$

with

$$\|R^\varepsilon\|_{L^\infty(L^2)} = \mathrm{O}\big(\varepsilon^{1/2}\big), \qquad \|r^\varepsilon\|_{W^{1,\infty}(L^2)} = \mathrm{O}\big(\varepsilon^{3/2}\big), \qquad \|r^\varepsilon\|_{W^{1,\infty}(H^1)} = \mathrm{O}\big(\varepsilon^{1/2}\big),$$

$$\|\sigma^\varepsilon\|_{L^\infty(H^{1/2}(\Sigma))} = \mathrm{O}(\varepsilon), \qquad \|\varphi^\varepsilon\|_{W^{1,\infty}(H^{1/2}(\Gamma^\varepsilon))} = \mathrm{O}\left(\exp\left(-\frac{\lambda}{\varepsilon}\right)\right), \quad \lambda > 0.$$

Boundary term φ^ε is the trace of the upper (respectively lower) boundary layer on the lower (respectively upper) boundary. This explains the exponential bound on its norm. We must add a corrector to v^ε to correct the divergence and boundary conditions.

5.2.1. Lift of the divergence and boundary terms

As supposed in Section 1, boundaries $\Gamma_1^\varepsilon := \Gamma_1^\varepsilon(x,y)$ and $\Gamma_2^\varepsilon := \Gamma_2^\varepsilon(x,y)$ are Lipschitz functions, 1-periodic in (x,y): for all x, y, x', y',

$$\left|\Gamma_j^\varepsilon(x',y') - \Gamma_j^\varepsilon(x,y)\right| \leqslant \frac{\mathrm{Lip}(\gamma_j)}{\varepsilon}\big(|x'-x| + |y'-y|\big).$$

Moreover, there exists $\alpha > 0$ such that for all x, y, $|\Gamma_j^\varepsilon(x,y)| \geqslant \alpha\varepsilon$. In particular, we have:

$$\left|\frac{1}{\Gamma_j^\varepsilon(x',y')} - \frac{1}{\Gamma_j^\varepsilon(x,y)}\right| \leqslant \frac{\mathrm{Lip}(\gamma_j)}{\varepsilon^3}\big(|x'-x| + |y'-y|\big)$$

so that $1/\Gamma_j^\varepsilon$ is also Lipschitz. Note that it implies that Γ_j^ε and $1/\Gamma_j^\varepsilon \in H^1(\mathbb{R}^2)$. Now we consider:

$$w_1^\varepsilon(t,x,y,z) = \varphi^\varepsilon(t,x,y)\left(\frac{2z - \Gamma_1^\varepsilon(x,y)}{\Gamma_1^\varepsilon(x,y)}\chi_1(z) + \frac{2z - \Gamma_2^\varepsilon(x,y)}{\Gamma_2^\varepsilon(x,y)}\chi_2(z)\right)$$

with

$$\chi_1 \in \mathcal{C}_c^\infty\left(\left[-\frac{1}{2},\frac{1}{2}\right]\right), \quad \chi_1 = 1 \quad \text{on } \left[-\frac{1}{4},\frac{1}{4}\right],$$
$$\chi_2 \in \mathcal{C}_c^\infty\left(\left[\frac{1}{2},\frac{3}{2}\right]\right), \quad \chi_2 = 1 \quad \text{on } \left[\frac{3}{4},\frac{5}{4}\right].$$

From properties of Γ_j^ε, we deduce easily that

$$w_1^\varepsilon \in W^{1,\infty}\big(0,T;\ H^1(\Omega^\varepsilon)\big)^3, \quad \text{1-periodic in } (x,y), \quad w_1^\varepsilon|_{\Sigma_1\cup\Sigma_2} = \varphi^\varepsilon,$$

$$\left\|w_1^\varepsilon\right\|_{W^{1,\infty}(H^1)} \leqslant C\exp\left(-\frac{\lambda'}{\varepsilon}\right), \quad \lambda' > 0.$$

Now, we may apply Lemma 3.1 of [10], whose proof extends easily to our horizontal perodicity conditions: there exists $w_2^\varepsilon \in L^\infty(H_0^1(\Omega^\varepsilon))^3$ such that

$$\nabla\cdot w_2^\varepsilon = r^\varepsilon - \nabla\cdot w_1^\varepsilon, \quad \left\|w_2^\varepsilon\right\|_{L^\infty(H^1)} \leqslant C(\Omega^\varepsilon)\left\|r^\varepsilon - \nabla\cdot w_1^\varepsilon\right\|_{L^\infty(L^2)}.$$

Moreover, we can choose

$$C(\Omega^\varepsilon) = C\delta(\Omega^\varepsilon)^3\big(1 + \delta(\Omega^\varepsilon)\big) \leqslant C',$$

where δ is the Lebesgue measure on $\mathbb{R}^3$, and C' is independent of ε. A look at the proof also shows that the same inequality holds for time derivatives, so that finally,

$$\left\|w_2^\varepsilon\right\|_{W^{1,\infty}(H^1)} \leqslant C'\varepsilon^{3/2}.$$

If we set $v^\varepsilon = w^\varepsilon + w_1^\varepsilon + w_2^\varepsilon$, w^ε is solution of:

$$\begin{aligned}
&\partial_t w^\varepsilon + \frac{\mathbf{e} \times w^\varepsilon}{\varepsilon} + \frac{\nabla q^\varepsilon}{\varepsilon} - \varepsilon \Delta w^\varepsilon + w^\varepsilon \cdot \nabla u^\varepsilon_{\mathrm{app}} + u^\varepsilon \cdot \nabla w^\varepsilon = \widetilde{R}^\varepsilon \quad \text{in } \Omega \cup \Omega_i^\varepsilon,\\
&\nabla \cdot w^\varepsilon = 0 \quad \text{in } \Omega \cup \Omega_i^\varepsilon,\\
&[w^\varepsilon]|_{\Sigma_i} = 0 \quad \text{at } \Sigma_i,\\
&\left[\varepsilon \frac{\partial w^\varepsilon}{\partial n} - \varepsilon^{-1} q^\varepsilon \vec{n}\right]\bigg|_{\Sigma_i} = \tilde{\sigma}^\varepsilon \quad \text{at } \Sigma_i,\\
&w^\varepsilon = 0 \quad \text{at } \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon,
\end{aligned} \tag{5.7}$$

with

$$\widetilde{R}^\varepsilon = R_1^\varepsilon + R_2^\varepsilon, \quad \left\| R_1^\varepsilon \right\|_{L^\infty(L^2)} = \mathrm{O}\big(\varepsilon^{1/2}\big),$$
$$\left\| R_2^\varepsilon \right\|_{L^\infty(H^{-1})} = \mathrm{O}\big(\varepsilon^{3/2}\big), \ \|\tilde{\sigma}^\varepsilon\|_{L^\infty(H^{-1/2})} = \mathrm{O}(\varepsilon).$$

5.2.2. Final estimates

An energy estimate on w^ε gives:

$$\begin{aligned}
&\partial_t \left\| w^\varepsilon(t,\cdot) \right\|_{L^2}^2 + \varepsilon \left\| \nabla w^\varepsilon(t,\cdot) \right\|_{L^2}^2\\
&\quad \leqslant \int_\sigma \tilde{\sigma}^\varepsilon \cdot w^\varepsilon(t,\cdot) + \int_{\Omega^\varepsilon} R_1^\varepsilon \cdot w^\varepsilon(t,\cdot) + \int_{\Omega^\varepsilon} R_2^\varepsilon \cdot w^\varepsilon(t,\cdot) + \int_{\Omega^\varepsilon} \big(w^\varepsilon \cdot \nabla u^\varepsilon_{\mathrm{app}}\big) \cdot w^\varepsilon\\
&\quad = I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{5.8}$$

I_1 satisfies

$$\begin{aligned}
|I_1| &\leqslant \|\tilde{\sigma}^\varepsilon\|_{L^\infty(H^{-1/2})} \left\| w^\varepsilon(t,\cdot) \right\|_{H^{1/2}} \leqslant C\varepsilon \left\| \nabla w^\varepsilon(t,\cdot) \right\|_{L^2}\\
&\leqslant C^2 \varepsilon + \frac{\varepsilon}{4} \left\| \nabla w^\varepsilon(t,\cdot) \right\|_{L^2}^2.
\end{aligned} \tag{5.9}$$

Similarly, we get

$$|I_3| \leqslant \ C_1 \varepsilon + \varepsilon^4 \left\| \nabla w^\varepsilon(t,\cdot) \right\|_{L^2}^2.$$

I_2 is bounded by

$$|I_2| \leqslant C\sqrt{\varepsilon}\, \left\| w^\varepsilon(t,\cdot) \right\|_{L^2}.$$

I_4 is the most difficult to control, because the boundary layer part of $u^\varepsilon_{\mathrm{app}}$ has strong gradient. Indeed,

$$\nabla u^\varepsilon_{\mathrm{app}} = \varepsilon^{-1}\nabla_{\mathbf{X}}\tilde{u}^0\left(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z}{\varepsilon}\right) + \varepsilon^{-1}\nabla_{\mathbf{X}}\bar{u}^0\left(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z-1}{\varepsilon}\right)$$
$$+ \text{higher-order terms in } \varepsilon.$$

Focusing as usual on the bottom layer, the worst term is then

$$J^\varepsilon = \varepsilon^{-1}\int_{\Omega^\varepsilon}\left(w^\varepsilon \cdot \nabla_{\mathbf{X}}\tilde{u}^0\left(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z}{\varepsilon}\right)\right)w^\varepsilon.$$

Introducing the auxiliary function

$$G(X, Y, Z) = \sup_{x,y\in\mathbb{T}^2}\left|\nabla_X\tilde{u}^0(x, y, X, Y, Z)\right|,$$

we have:

$$|J^\varepsilon| \leqslant \varepsilon^{-1}\int_{\Omega^\varepsilon}|w^\varepsilon|^2 G\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z}{\varepsilon}\right)\mathrm{d}\mathbf{x}.$$

We are left with the control of

$$\begin{aligned}
I_\varepsilon &= \int_{\Omega^\varepsilon}|w^\varepsilon|^2 G\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z}{\varepsilon}\right)\mathrm{d}\mathbf{x}\\
&= \int_{\Omega^\varepsilon\cap\{z>R\varepsilon\}}|w^\varepsilon|^2 G\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z}{\varepsilon}\right)\mathrm{d}\mathbf{x} + \int_{\Omega^\varepsilon\cap\{z<R\varepsilon\}}|w^\varepsilon|^2 G\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z}{\varepsilon}\right)\mathrm{d}\mathbf{x}\\
&= I_R^{\varepsilon,+} + I_R^{\varepsilon,-},
\end{aligned}$$

where $R > 0$ will be chosen below.

Study of $I_R^{\varepsilon,+}$. Thanks to Corollary 3.8, there exists $R_1, C, \sigma > 0$ such that for all $Z > R_1$,

$$G(X, Y, Z) \leqslant C\exp(-\sigma Z). \tag{5.10}$$

Hence, for all $\delta > 0$, there exists $R = R(\delta) > R_1$ such that

$$\sup_{X,Y,Z\geqslant R} Z^2 G(X, Y, Z) \leqslant \delta.$$

Then we may write:

$$I_R^{\varepsilon,+} \leqslant \varepsilon^2\Big(\sup_{X,Y,\ Z\geqslant R} Z^2 G(X, Y, Z)\Big)\int_{\Omega^\varepsilon\cap\{z>R\varepsilon\}}\left|\frac{w^\varepsilon}{z}\right|^2 \leqslant C\delta\varepsilon^2\|\nabla w^\varepsilon\|^2_{L^2},$$

where we have used Hardy's inequality. Chosing $\delta = 1/4C$ and $R = R(\delta)$, it leads to

$$I_R^{\varepsilon,+} \leqslant \frac{\varepsilon^2}{4} \|\nabla w^\varepsilon\|_{L^2}^2.$$

Study of $I_R^{\varepsilon,-}$. It remains to bound

$$I_R^{\varepsilon,-} = \int\limits_{\Omega^\varepsilon \cap \{z \leqslant R\varepsilon\}} w^\varepsilon(x,y,z)^2 G\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{z}{\varepsilon}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

We can write:

$$\Omega^\varepsilon \cap \{z \leqslant R\varepsilon\} = \bigcup_{k \in K \subset \mathbb{Z}^2} \varepsilon(\omega + k), \quad \operatorname{card}(K) = \frac{n}{\varepsilon^2}$$

for a domain ω. Now,

$$\begin{aligned} I_R^{\varepsilon,-} &= \varepsilon^3 \sum_{k \in K} \int\limits_{\omega+k} \left| w^\varepsilon(\varepsilon X, \varepsilon Y, \varepsilon Z)\right|^2 G(X,Y,Z)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z \\ &\leqslant \varepsilon^3 \sum_{k \in K} \|G\|_{L^2(\omega+k)} \left(\int\limits_{\omega+k} \left| w^\varepsilon(\varepsilon X, \varepsilon Y, \varepsilon Z)\right|^4 \mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z \right)^{1/2}. \end{aligned}$$

As F is periodic,

$$I_R^{\varepsilon,-} \leqslant \varepsilon^3 \|G\|_{L^2(\omega)} \sum_{k \in K} \|\widetilde{w}\|_{L^4(\omega+k)}^2,$$

where $\widetilde{w}(X,Y,Z) = w^\varepsilon(\varepsilon X, \varepsilon Y, \varepsilon Z)$. By Sobolev imbedding

$$\|\widetilde{w}\|_{L^4(\omega+k)}^2 \leqslant C \|\nabla \widetilde{w}\|_{H^1(\omega+k)}^2,$$

where C is independent of k. So

$$I_R^{\varepsilon,-} \leqslant C\varepsilon^3 \|G\|_{L^2(\widetilde{\omega})} \int\limits_{\omega+k} \left|\nabla_{\mathbf{X}} \widetilde{w}(X,Y,Z)\right|^2 \mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z \leqslant C\varepsilon^2 \|G\|_{L^2(\widetilde{\omega})} \|\nabla w^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2.$$

We thus have

$$I_R^{\varepsilon,-} \leqslant C'\varepsilon^2 \sup_{x,y \in \mathbb{T}^2} \left\|\nabla \tilde{u}^0\right\|_{L^2(\widetilde{\omega})} \left\|\nabla w^\varepsilon(t,\cdot)\right\|_{L^2}^2 \leqslant C''\varepsilon^2 \left(\sup |u^0|\right) \left\|\nabla w^\varepsilon(t,\cdot)\right\|_{L^2}^2,$$

where we have used the estimate (4.7). Then, assuming that $\sup|u^0|$ is small, it yields

$$I_R^{\varepsilon,-} \leqslant \frac{\varepsilon^2}{4}\left\|\nabla w^\varepsilon(t,\cdot)\right\|_{L^2}^2.$$

Conclusion. Under a smallness assumption on $\sup|u^0|$ (which if fulfilled under a smallness assumption on $\|u_0\|_{L^\infty}$), we can absorb the squares of the L^2 norm of ∇w^ε at the right-hand side of (5.8) into the diffusive term $\varepsilon\|\nabla w^\varepsilon(t,\cdot)\|^2$ at the left-hand side. We find inequality of type

$$\partial_t\left\|w^\varepsilon(t,\cdot)\right\|_{L^2}^2 + \frac{\varepsilon}{2}\left\|\nabla w^\varepsilon(t,\cdot)\right\|_{L^2}^2 \leqslant C(\varepsilon) + D\left\|w^\varepsilon(t,\cdot)\right\|_{L^2}^2$$

with $C(\varepsilon) \to 0$ as $\varepsilon \to 0$. We conclude thanks to a Gronwall Lemma.

5.3. Generalization

5.3.1. Other types of roughness

We have considered rough boundaries of type $\Gamma := \varepsilon\Gamma(\varepsilon^{-1}x, \varepsilon^{-1}y)$. We can consider more general boundaries of type

$$\Gamma := \varepsilon\Gamma\big(x, y, \varepsilon^{-1}\varepsilon^{-1}y\big).$$

Boundary layer terms $\tilde{u}^i(t,x,y,X,Y,Z)$ satisfy in this case: $\forall t,x,y$, $\tilde{u}^i(t,x,y,\cdot)$ is defined in $\widetilde{\omega}_{x,y}$ where $\widetilde{\omega}_{x,y}$ is the boundary layer domain associated to $\widetilde{\Gamma}_{x,y} = \widetilde{\Gamma}(x,y,\cdot)$. Proceeding as in this paper, all the results adapt to these rough domains.

5.3.2. Physical insight

In our view, the mathematical study performed in this paper has real physical insight, and is a good basis for further investigations. For instance, we have shown that the variations of the kinetic energy of the fluid obey to equation

$$\frac{\mathrm{d}}{2\mathrm{d}t}\left\|u(t,\cdot)\right\|_{L^2}^2 = -\int\limits_{\mathbb{T}^2} P\big(u(t,\cdot)\big)\cdot u(t,\cdot),$$

with a function P being intricate but explicit. Therefore, it is now possible to understand how the energy dissipation depends on the roughness: it will be the matter of a forthcoming paper. Another relevant problem is the way the roughness affects the stability of the boundary layers. Now that we have derived the equations of the layers, one is able to make quantitative studies (critical Reynolds number, ...). Moreover, it is possible to apply the same type of analysis to MHD systems, for instance with regards to geophysical issues.

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